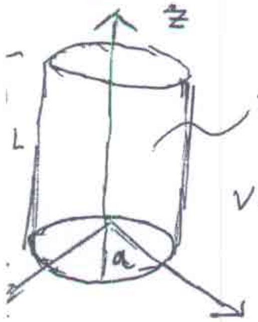
Solving for the ~~Wavefunction~~ in Cylindrical Coordinates (Particle in a Cylindrical Box)

Energy of the Wavefunction



V=0 inside the cylinder

V=∞ outside the cylinder with radius a, and height L

'n the TISE inside the cylinder is:

m_e = mass of an electron (Let's just say our particle has this mass)

$$\frac{p^2}{2m}\psi = E\psi \;;\quad \frac{-\hbar^2}{2m_e}\nabla^2\psi = E\psi$$

$$\nabla^2\psi + \frac{2mE}{\hbar^2}\psi = 0 \;;\ \text{Call } \frac{2m_eE}{\hbar^2} = k^2$$

$$\therefore\ \nabla^2\psi + k^2\psi = 0$$; This is just the Helmholtz' Eqn.

We can use separation of variables to break this into 3 ordinary D.E. such that $\psi = R(\rho)\,Q(\phi)\,Z(z)$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \;;\quad \frac{d^2 Z}{dz^2} + \alpha^2 Z = 0$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \;;$$

The Z & Q solutions are elementary

i.e. $Z(z) = C_1 e^{i\alpha z} + C_2 e^{-i\alpha z}$

$Q(\phi) = C_3 e^{im\phi} + C_4 e^{-im\phi}$

However the radial portion may be unfamiliar; The solutions to these type of differential eqns. are the power series known as <u>Bessel Functions</u>.

The Particular solutions to this equation are:

$J_m(\sqrt{k^2-\alpha^2}\,\rho)$ & $N_m(\sqrt{k^2-\alpha^2}\,\rho)$ such that

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{m+2n} \;\&\; N_m(x) = \frac{\cos m\pi\, J_m(x) - J_{-m}(x)}{\sin m\pi}$$

:z) ≡ (z-1)!; ...en z is an ...teger.

An aside $J_m(x)$ are known as Bessel Functions of the 1st kind & $N_m(x)$ are called Neumann functions (Also denoted $Y_m(x)$ in older texts)

Now in order for our Ψ to be singly valued, the following must be true

$$e^{im\phi} = e^{im(\phi+2\pi)} \;;\; \therefore m = 0,1,2\ldots$$

$\boxed{\text{* } a_m \text{ and } b_m \text{ are just constants.}}$

$$\therefore \Psi(z,\phi,\rho) = \sum_{m=0}^{\infty}{}^{*} (a_m \cos m\phi + b_m \sin m\phi)(c_1 J_m + c_2 N_m)(c_3 \cos \alpha z + c_4 \sin \alpha z)$$

Note: Ψ must be finite at $\rho = 0$, $\therefore c_2 = 0$ (Due to J_{-m} part blowing up)

Also: Ψ must be $=0$ at $z=0$, $\therefore$ $c_3 = 0$

$=0$ at $z = L$, $\therefore$ $\sin \alpha L = 0$ $\quad \alpha L = n\pi$

$\therefore \alpha = \frac{n\pi}{L}$; $n = 0,1,2\ldots$

$$\therefore \Psi(z,\phi,\rho) = \sum_{m,n}^{\infty} (a_m \cos m\phi + b_m \sin m\phi)\left(J_m\left(\sqrt{k^2 - \frac{n^2\pi^2}{L^2}}\,\rho\right)\right)\left(\sin \frac{n\pi}{L} z\right)$$

$$\Psi(z,\phi,a) = 0 \;;\; \therefore J_m\left(\sqrt{k^2 - \frac{n^2\pi^2}{L^2}}\,a\right) = 0$$

(Since ϕ and z portions don't ~~...~~ have to ~~...~~ disappear).

In other words $\sqrt{k^2 - \frac{n^2\pi^2}{L^2}}\,a = C_{m,n}$ where $C_{m,n}$ are the zeros of the Bessel functions (You can look these up in various tables, such as in Abramowitz & Stegun, but they aren't necessary to solve this problem.

$$\therefore \quad C_{m,n}^{\,2} = a^2\left(\frac{2m_e E}{\hbar^2} - \frac{n^2\pi^2}{L^2}\right)$$

$$\left((C_{m,n}^{\,2}) + \frac{a^2 n^2 \pi^2}{L^2}\right) = a^2 \frac{2m_e E}{\hbar^2}$$

$$\frac{\hbar^2}{2m_e}\left(\frac{C_{m,n}^{\,2}}{a^2} + \frac{n^2\pi^2}{L^2}\right) = E_{m,n}$$

(This boundary condition has now quantized our energy; qed)

So we are left w/

$$\Psi = \sum_{m,n} (a_m \cos m\phi + b_m \sin m\phi)\left(J_m \sqrt{k^2 - \frac{n^2\pi^2}{L^2}}\,\rho\right)\left(\sin \frac{n\pi}{L} z\right)$$

and $E_{m,n} = \frac{\hbar^2}{2m_e}\left(\frac{C_{m,n}^{\,2}}{a^2} + \frac{n^2\pi^2}{L^2}\right)$; qed (Note: In order to solve for the a_m & b_m we would need additional B.C.'s)

Useful Properties of the Bessel Functions

limit $x << 1$

$$J_m(x) \approx \frac{1}{\Gamma(m+1)} \left(\frac{x}{2}\right)^m$$

$$N_m(x) \approx \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \cdots\right] \; m=0 \; ; \; -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m \; m \neq 0$$

limit $x >> 1$

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$N_m(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

Very Basic Magnetostatics (For the HW)

$\vec{B} = \vec{\nabla} \times \vec{A}$ where $\vec{A}$ is the Vector Potential

$\therefore H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$ of a charge particle in a vector potential

$$\Downarrow$$

$$H = \frac{1}{2m}\left(p^2 - \left(\frac{e}{c}\right)(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}) + \left(\frac{e}{c}\right)^2 \vec{A}^2\right)$$

However Note that this simply translates to (I say simply because you need not know how to do it)

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{z}\frac{\partial}{\partial z} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} \quad (\text{No } \vec{B}\text{-field})$$

$$\Downarrow$$

$$\vec{\nabla} = \quad '' \quad '' \quad + \hat{\phi}\frac{1}{\rho}\left(\frac{\partial}{\partial\phi} - \left(\frac{ie}{\hbar c}\right)\frac{B\rho_a^2}{2}\right)$$

{when you have a magnetic field such as in Problem 25.}

for $\rho_a < \rho < \rho_b$