$$\langle j,m | J_+^\dagger J_+ | j,m\rangle = |c_{jm}^+|^2 \underbrace{\langle j,m+1 | j,m+1\rangle}_{=1}$$

and $J_+^\dagger J_+ = J^2 - J_z^2 - \hbar J_z$ so

$$\langle j,m | J^2 - J_z^2 - \hbar J_z | j,m\rangle = \hbar^2 j(j+1) - \hbar^2 m^2 - \hbar^2 m$$

$$\Rightarrow |c_{jm}^+|^2 = \hbar^2 [j(j+1) - m^2 - m] = \hbar^2 [j(j+1) - m(m+1)]$$
$$= \hbar^2 (j-m)(j+m+1)$$

By convention we can choose c_{jm}^+ to be real and positive:

$c_{jm}^+ = \hbar\sqrt{(j-m)(j+m+1)}$ thus

$$\boxed{J_+ |j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}\, |j,m+1\rangle}$$

Similarly we can show that

$$\boxed{J_- |j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\, |j,m-1\rangle}$$

Now how do we compute the rotation operator for various values of j?

$$\mathcal{D}^{(j)}_{m'm}(\hat{n},\varphi) = \langle j,m' | e^{-i\vec{J}\cdot\hat{n}\varphi/\hbar} | j,m\rangle \quad \text{Wigner functions}$$

Note: $[\mathcal{D}(\hat{n},\varphi), J^2] = 0$ since $[J^2, J_x] = 0$ so

$\mathcal{D}$ can scramble up the m's but cannot mix the j's!

Lets think about the full rotation operator ← capable of rotating *any* ket!

all j's

$$\mathcal{D}_{m'm} = \begin{array}{c} \\ \\ \\ \end{array}\begin{pmatrix} \boxed{j=1/2} & 0 & 0 & 0 \\ 0 & \boxed{j=1} & 0 & 0 \\ 0 & 0 & \boxed{j=3/2} & \\ & & & \ddots \end{pmatrix}\begin{array}{l} 2 \\ 3 \\ 4 \\ \\ \end{array}$$

acts on $j=3/2$ states.

necessary to prevent mixing

There is some representation in which the rotation operator breaks up into this block diagonal form ← we will call these <u>irreducible representations</u> of a rotation group.

Note that each irreducible representation forms its own subgroup.

$$\Longrightarrow \quad \sum_{m'} \mathcal{D}^{(j)}_{m''m'}(R_1)\,\mathcal{D}^{(j)}_{m'm}(R_2) = \mathcal{D}^{(j)}_{m''m}(\underbrace{R_1R_2}_{\text{single rotation}})$$

Since the $\mathcal{D}$'s are unitary operators

$$\mathcal{D}^{(j)}_{m'm}(R^{-1}) = \mathcal{D}^{(j)*}_{mm'}(R)$$

You know what is the identity! ↑ Hermitian conjugate matrix element.

Additionally, we know how to rotate a state w/ definite j,m:

$$\mathcal{D}(R)\,|j,m\rangle = \sum_{m'} |j,m'\rangle\langle j,m'|\,\mathcal{D}(R)\,|j,m\rangle$$

$$= \sum_{m'} \mathcal{D}^{(j)}_{m'm}(R)\,|j,m'\rangle$$

Wigner functions.

Now, it remains to calculate these functions.

Recall our discussion of Euler Angle characterization of an arbitrary rotation

$$\mathcal{D}^{(j)}_{m'm}(\underbrace{\alpha\beta\gamma}_{\text{three angles.}}) = \langle j,m'| e^{-iJ_z\alpha/\hbar} e^{-iJ_y\beta/\hbar} e^{-iJ_z\gamma/\hbar} |j,m\rangle$$

$$= e^{-i(m'\alpha+m\gamma)} \underbrace{\langle j,m'| e^{-iJ_y\beta/\hbar} |j,m\rangle}_{\text{only one complicated part to do.}}$$

Define $d^{(j)}_{m'm}(\beta) = \langle j,m'| e^{-iJ_y\beta/\hbar} |j,m\rangle$

We already worked this out for $j=1/2$

$$d^{(1/2)}_{m'm}(\beta) = \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix}$$

How do we do this again? Recall $J_y = \dfrac{J_+ - J_-}{2i}$

so for $j=1$ we need to calculate

$$\langle 1,m'| J_y |1,m\rangle$$

$$= \frac{\hbar}{2} \begin{array}{cl} \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix} & \begin{array}{l} \leftarrow m'=1 \\ \leftarrow 0 \\ \leftarrow -1 \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 1 & 0 & -1 \end{array}\ m & \end{array}$$

The diagonals are easy. Lets try $\langle 1,0|J_y|1,1\rangle$

$$\langle 1,0| \frac{J_+ - J_-}{2i} |1,1\rangle = -\frac{1}{2i}\langle 1,0|J_-|1,1\rangle = -\frac{\hbar}{2i}\sqrt{(1+1)(1-1+1)}$$

$$= \frac{\sqrt{2}i\hbar}{2}$$

Since J_y is Hermitian, we only need to compute one more...

But we're not out of the woods yet! We still need to work at J_y^n for all $n \geq 0$.

$$\left(J_y^{(j=1)}\right)^2 = \frac{\hbar^2}{4}\begin{pmatrix}0 & -\sqrt{2}i & 0\\ \sqrt{2}i & 0 & -\sqrt{2}i\\ 0 & \sqrt{2}i & 0\end{pmatrix}\begin{pmatrix}0 & -\sqrt{2}i & 0\\ \sqrt{2}i & 0 & -\sqrt{2}i\\ 0 & \sqrt{2}i & 0\end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}2 & 0 & -2\\ 0 & 4 & 0\\ -2 & 0 & 2\end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix}1 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{pmatrix}$$

Now look at this operator cubed:

$$\left(J_y^{(j=1)}\right)^3 = \frac{\hbar^2}{2}\begin{pmatrix}1 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -\sqrt{2}i & 0\\ \sqrt{2}i & 0 & -\sqrt{2}i\\ 0 & \sqrt{2}i & 0\end{pmatrix} = \frac{\hbar^3}{4}\begin{pmatrix}0 & -2\sqrt{2}i & 0\\ 2\sqrt{2}i & 0 & -2\sqrt{2}i\\ 0 & 2\sqrt{2}i & 0\end{pmatrix}$$

so $\left(J_y^{(j=1)}\right)^3 = \hbar^2 J_y^{(j=1)}$ so they do repeat after a couple.

$$e^{-iJ_y\beta/\hbar} = 1 - iJ_y\beta/\hbar + \frac{\beta^2}{2!}\left(\frac{J_y}{\hbar}\right)^2 + \frac{\beta^3}{3!}i\left(\frac{J_y}{\hbar}\right)^3 + \frac{\beta^4}{4!}\left(\frac{J_y}{\hbar}\right)^4$$

For $j=1$ only

$$= 1 - \left(\frac{J_y}{\hbar^2}\right)^2\left\{\frac{\beta^2}{2!} - \frac{\beta^4}{4!} + \frac{\beta^6}{6!} - \cdots\right\} +$$

$$- \frac{iJ_y}{\hbar}\left\{\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \cdots\right\}$$

So $(j=1)$ $e^{-iJ_y\beta/\hbar} = 1 - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta) - \frac{iJ_y}{\hbar}\sin\beta$

Now, we can compute $d^{(1)}(\beta)$:

$$d^{(1)}_{m'm}(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2}(1-\cos\beta)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \sin\beta \begin{pmatrix} 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$d^{(1)}_{m'm}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

O.k. But this seems like a complicated way to do this. Later, we will find an easier way.

Let's jump to looking at how to consider rotations of a spatial wavefunction. finite-dimensional representations

Consider the wavefunction $\langle \vec{x} | \psi \rangle$. How do we rotate it?

$$\mathcal{D}(\hat{n},\varphi)|\vec{x}\rangle = |\vec{x}'\rangle \quad \text{where}$$
$$\vec{x}' = \overleftrightarrow{R}(\hat{n},\varphi)\cdot\vec{x}$$

orthogonal rotation matrix.

So
$$\langle \vec{x} | \mathcal{D}(\hat{n},\varphi) | \psi \rangle = \langle \psi | \mathcal{D}^{+}(\hat{n},\varphi) | \vec{x} \rangle^{*} = \langle \psi | R^{-1}\cdot\vec{x} \rangle^{*}$$

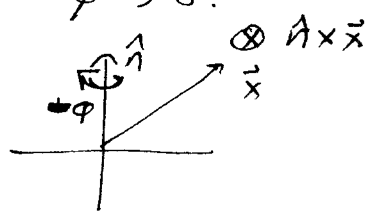
unitary operator so
$$\mathcal{D}^{+}(\hat{n},\varphi) = \mathcal{D}^{-1}(\hat{n},\varphi)$$

$\Longrightarrow$
$$\langle \vec{x} | \mathcal{D}(\hat{n},\varphi) | \psi \rangle = \langle R^{-1}\cdot\vec{x} | \psi \rangle$$

rotate the $\vec{x}$ bra the other way!

Consider an infinitesimal rotation: i.e. $\varphi \to 0$.

Then $\overleftrightarrow{R}^{-1}\cdot\vec{x} = \vec{x} - \varphi\hat{n}\times\vec{x}$;



Thus: $\langle \vec{x} | e^{-i\vec{J}\cdot\hat{n}\varphi/\hbar} | \psi\rangle \xrightarrow[\varphi\to 0]{} \langle \vec{x} | \psi\rangle - \frac{i\varphi}{\hbar}\langle \vec{x} | \vec{J}\cdot\hat{n} | \psi\rangle$

$$= \langle \vec{x} - \varphi\hat{n}\times\vec{x} | \psi\rangle = \psi(\vec{x}) - \varphi\, n_\alpha x_\beta \varepsilon_{\alpha\beta k}\frac{\partial\psi}{\partial x_k}$$

$$\Rightarrow \langle \vec{x} | J_\ell | \psi\rangle = -i\hbar\, \varepsilon_{\ell\beta k} x_\beta \frac{\partial\psi}{\partial x_k} \Rightarrow$$

$$\langle \vec{x} | \vec{J} | \psi\rangle = \vec{x}\times\left(-i\hbar\frac{\partial}{\partial\vec{x}}\right)\psi(x) \quad \text{or}$$

$\vec{J} = \vec{x}\times\vec{p}$ ← This is just the classical definition of angular momentum.

We will call it $\vec{L}$ to distinguish it from $\vec{S}$ — spin angular momentum.

$$\boxed{\vec{L} = \vec{r}\times\vec{p}}$$

Let's look at the components:

$\underline{\underline{L_z}}$: Infinitesimal rotation about $\hat{z}$: $1 - \frac{i\delta\varphi}{\hbar}L_z = 1 - \frac{i\delta\varphi}{\hbar}(xp_y - yp_x)$

acts on a ket $|x,y,z\rangle$ — $i\hbar\partial_y$ $+ i\hbar\partial_x$ ← when acting on the ket

$$\left(1 - \frac{i\delta\varphi}{\hbar}L_z\right)|x,y,z\rangle = \left[1 - \frac{ip_y}{\hbar}\delta\varphi x + \frac{ip_x}{\hbar}\delta\varphi y\right]|x,y,z\rangle$$

$$= |x - \delta\varphi y, y + x\delta\varphi, z\rangle$$

$\Rightarrow$

$$\langle x,y,z | \left[1 - \frac{i\delta\varphi L_z}{\hbar}\right] | \alpha\rangle = \langle x + \delta\varphi y, y - x\delta\varphi, z | \alpha\rangle$$

What is this in spherical coordinates.

$$x = r\sin\theta\cos\varphi \quad y = r\sin\theta\sin\varphi \quad z = r\cos\theta$$

$$x \to x + \delta\varphi y = r\sin\theta[\cos\varphi + \delta\varphi\sin\varphi] = r\sin\theta\cos(\varphi - \delta\varphi)$$

$y \to y - x\delta\varphi = \quad r\sin\theta\left[\sin\varphi - \delta\varphi\cos\varphi\right] = r\sin\theta\sin(\varphi-\delta\varphi)$

$z \to z \qquad\qquad r\cos\theta \to r\cos\theta$

$\Rightarrow$ r, θ fixed $\quad \varphi \to \varphi - \delta\varphi$

or

$$\langle r,\theta,\varphi|\left[1 - \frac{i\delta\varphi}{\hbar}L_z\right]|\alpha\rangle = \langle r,\theta,\varphi-\delta\varphi|\alpha\rangle$$

$$= \langle r,\theta,\varphi|\alpha\rangle - \delta\varphi\frac{\partial}{\partial\varphi}\langle r,\theta,\varphi|\alpha\rangle$$

matching terms

$\Rightarrow$

$$\langle\vec{x}|L_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\varphi}\langle\vec{x}|\alpha\rangle$$

Can do the same for L_x and L_y: (Homework)

$$\langle\vec{x}|L_x|\alpha\rangle = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\langle\vec{x}|\alpha\rangle$$

$$\langle\vec{x}|L_y|\alpha\rangle = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\langle\vec{x}|\alpha\rangle$$

Combining these we find:

$$\langle\vec{x}|L_\pm|\alpha\rangle = -i\hbar e^{\pm i\varphi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right)\langle\vec{x}|\alpha\rangle$$

Using $\quad L^2 = L_z^2 + \frac{1}{2}\left(L_+L_- + L_-L_+\right)$ we find:

$$\langle\vec{x}|L^2|\alpha\rangle = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\langle\vec{x}|\alpha\rangle$$

Just $r^2\nabla^2\Big|_{|\hat{r}|=1}$ — angular part of the Laplacian!

Why does this make sense?

$$L^2 = L_iL_i = \varepsilon_{i\alpha\beta}x_\alpha p_\beta\,\varepsilon_{i\delta\gamma}x_\delta p_\gamma$$

Note we have to be careful since $\vec{x}$ and $\vec{p}$ are <u>noncommuting</u> variables!

$$\varepsilon_{i\alpha\beta}\varepsilon_{i\delta\gamma} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}$$

$$L^2 = \left(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}\right)\left(x_\alpha p_\beta x_\delta p_\gamma\right) = x_\alpha\left(\delta_{\alpha\delta}\delta_{\beta\gamma}\right)p_\beta x_\delta p_\gamma - x_\alpha p_\beta\,\delta_{\alpha\gamma}\delta_{\beta\delta}\,x_\delta p_\gamma$$

Use the $[\vec{x}, \vec{p}]$ commutator (3.24)

$$L^2 = x_\alpha \delta_{\alpha\delta} \delta_{\beta\gamma} (x_\delta p_\beta - i\hbar \delta_{\delta\beta}) p_\gamma - \delta_{\alpha\gamma} \delta_{\beta\delta} x_\alpha p_\beta (p_\gamma x_\delta + i\hbar \delta_{\delta\gamma})$$

$$L^2 = x^2 p^2 - \vec{x}\cdot\vec{p}\, i\hbar - \delta_{\alpha\gamma} \delta_{\beta\delta} x_\alpha p_\gamma (x_\delta p_\beta - i\hbar \delta_{\beta\delta}) +$$
$$- \delta_{\alpha\gamma} \delta_{\beta\delta} x_\alpha p_\beta\, i\hbar \delta_{\delta\gamma}$$

$$L^2 = x^2 p^2 - i\hbar\, \vec{x}\cdot\vec{p} - (\vec{x}\cdot\vec{p})^2 + i\hbar\, \delta_{\delta\delta}\, \vec{x}\cdot\vec{p} - i\hbar\, \delta_{\alpha\delta} \delta_{\beta\delta} x_\alpha p_\beta$$

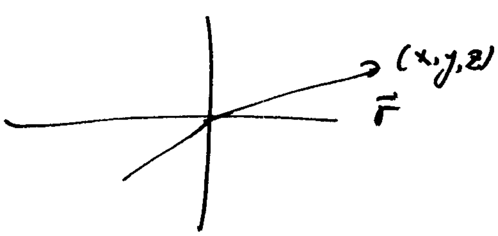
$$L^2 = x^2 p^2 - i\hbar\, \vec{x}\cdot\vec{p} - (\vec{x}\cdot\vec{p})^2 + 3 i\hbar\, \vec{x}\cdot\vec{p} - i\hbar\, \vec{x}\cdot\vec{p}$$

$$L^2 = x^2 p^2 + i\hbar\, \vec{x}\cdot\vec{p} - (\vec{x}\cdot\vec{p})^2$$

we would lose this term if we treated $\vec{x}$ and $\vec{p}$ as classical, commuting variables.

Now:

$$\langle \vec{x} | \vec{x}\cdot\vec{p} | \alpha \rangle = \vec{x} \cdot \langle \vec{x} | \vec{p} | \alpha \rangle = \vec{x}\cdot(-i\hbar \vec{\nabla}) \langle \vec{x} | \alpha \rangle$$
$$= -i\hbar\, r \frac{\partial}{\partial r} \langle \vec{x} | \alpha \rangle$$



Just apply $\vec{x}\cdot\vec{p}$ twice!

$$\langle \vec{x} | (\vec{x}\cdot\vec{p})^2 | \alpha \rangle = -\hbar^2 \left(r \frac{\partial}{\partial r} \right) \left(r \frac{\partial}{\partial r} \langle \vec{x} | \alpha \rangle \right)$$

$$= -\hbar^2 \left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} \right) \langle \vec{x} | \alpha \rangle$$

so

$\vec{x} p^2$ term

$$\langle \vec{x} | L^2 | \alpha \rangle = r^2 \langle \vec{x} | p^2 | \alpha \rangle + \hbar^2 \left[r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} \right] \langle \vec{x} | \alpha \rangle +$$

$$+ \hbar^2 r \frac{\partial}{\partial r} \langle \vec{x} | \alpha \rangle = r^2 \langle \vec{x} | p^2 | \alpha \rangle + r^2 \hbar^2 \left[\partial_r^2 + \frac{2}{r} \partial_r \right] \langle \vec{x} | \alpha \rangle$$

so

$$\underbrace{\langle \vec{x} | \frac{p^2}{2m} | \alpha \rangle}_{\text{kinetic energy}} = \frac{-\hbar^2}{2m} \nabla^2 \langle \vec{x} | \alpha \rangle = \text{next page....}$$

$$\langle \vec{x}|\frac{p^2}{2m}|\alpha\rangle = \frac{-\hbar^2}{2m}\nabla^2\langle \vec{x}|\alpha\rangle = \frac{1}{2mr^2}\langle \vec{x}|L^2|\alpha\rangle - \frac{\hbar^2}{2m}\left[\partial_r^2+\frac{2}{r}\partial_r\right]\langle x|\alpha\rangle$$

or
$$\langle \vec{x}|\frac{p^2}{2m}|\alpha\rangle = \frac{-\hbar^2}{2m}\left[\underbrace{\partial_r^2+\frac{2}{r}\partial_r}_{\text{radial part of the Laplacian}} - \underbrace{\frac{1}{\hbar^2 r^2}\langle \vec{x}|L^2|\alpha\rangle}_{\text{must be the angular part of the Laplacian!}}\right]$$

Spherical Harmonics:

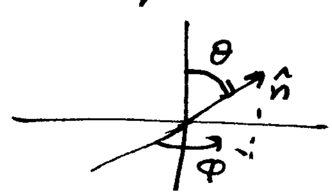
A spinless particle in a spherically symmetric potential $V=V(|\vec{r}|)$ has a wavefunction that is separable in spherical coordinates so that the energy eigenstates have the form:

$$\langle \vec{x}|n,l,m\rangle = R_{nl}(r)\,Y_l^m(\theta,\varphi)$$

Since $[H,L^2]=0$, $[H,L_z]=0$ the energy eigenkets are simultaneous eigenkets of L^2 and L_z with eigenvalues $l(l+1)\hbar^2$ and $m\hbar$ respectively. Recall $-l<m<l$, $m\in\mathbb{Z}$.

We can isolate the angular dependence by writing $\langle \hat{n}|l,m\rangle$

$$\langle \hat{n}|\theta,\varphi\rangle = Y_l^m(\theta,\varphi)$$



$|\hat{n}\rangle$ = "direction eigenket."

$Y_l^m(\theta,\varphi)$: amplitude for $|l,m\rangle$ state to be found in the direction (θ,φ) $= \langle \hat{n}|l,m\rangle$

What is $Y_l^m(\theta,\varphi)$: $\quad L_z|l,m\rangle = m\hbar|l,m\rangle$

$$\Rightarrow \langle \hat{n}|L_z|l,m\rangle = m\hbar\langle \hat{n}|l,m\rangle \Rightarrow -i\hbar\frac{\partial}{\partial\phi}\langle \hat{n}|l,m\rangle = m\hbar\langle \hat{n}|l,m\rangle$$

$\exists\, b_{max}$ $\quad J_+|a, b_{max}\rangle = 0$ $\qquad J_- J_+ = J_x^2 + J_y^2 - i(J_y J_x - J_x J_y)$

$$J_- J_+ |a, b_{max}\rangle = (J^2 - J_z^2 - \hbar J_z)|a, b_{max}\rangle = 0$$

$$\Rightarrow a - b_{max}^2 - b_{max}\hbar = 0 \Rightarrow a = b_{max}(b_{max} + \hbar)$$

Similarly $J_-|a, b_{min}\rangle = 0$ $\qquad J_+ J_- = J^2 - J_z^2 + \hbar J_z$

$$\Rightarrow a = b_{min}(b_{min} - \hbar)$$

$$\Rightarrow b_{max} = -b_{min} \Rightarrow -b_{max} \leq b \leq b_{max}$$

$b_{max} = b_{min} + n\hbar$

$\Rightarrow b_{max} = n\hbar/2$ $\leftarrow$ normally call this j.

$\boxed{j = n/2}$ $\rightarrow$ $a = \hbar^2 j(j+1)$

$b = m\hbar$ $\qquad m = \underbrace{-j, \ldots, j}_{2j+1 \text{ states.}}$

So we now see:

$$J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$$

$$J_z|j,m\rangle = m\hbar|j,m\rangle$$

How can we get all the states $|j,m\rangle$ for $-j \leq m \leq j$?

Ans: Apply the raising/lowering operators:

We need to find $J_+|j,m\rangle = c_{jm}^+|j,m+1\rangle$

To get the normalization constant we compute:

so $\langle \hat{n} | l,m \rangle \sim e^{+im\varphi}$ What about the θ-dependence?

$$\langle \hat{n} | L^2 | l,m \rangle = l(l+1)\hbar^2 | l,m \rangle$$ Partial Differential Equation for Y_l^m

$$\Longrightarrow \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + l(l+1)\right] Y_l^m = 0$$

$$\langle l',m' | l,m \rangle = \delta_{ll'}\delta_{mm'} \Longrightarrow$$

$$\int d\hat{n} \, \underbrace{\langle l',m' | \hat{n} \rangle \langle \hat{n} | l,m \rangle}_{\text{completeness.}} = \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta) \, Y_{l'}^{m'*}(\theta,\varphi) \, Y_l^m(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

How do we get the Y_l^m's? Ans. We can solve a lower order PDE instead of the above.

Consider $m=l$ $\quad L_+ | l,l \rangle = 0$

$$= -i\hbar e^{i\varphi}\left[i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]\langle \hat{n} | l,m \rangle = 0$$

But $\langle \hat{n}, | l,m \rangle \sim e^{il\varphi} \Rightarrow$

$$\cancel{i}\frac{\partial}{\partial\theta}\langle \hat{n} | l,m \rangle = \cot\theta \, \cancel{i} l \, \langle \hat{n} | l,m \rangle$$

$$\tan\theta\frac{\partial}{\partial\theta} = \frac{\sin\theta}{\cos\theta}\frac{\partial}{\partial\theta} = ?$$

$$\Rightarrow \quad \frac{\partial}{\partial\theta} = \frac{\partial\sin\theta}{\partial\theta}\frac{\partial}{\partial\sin\theta} = \cos\theta\frac{\partial}{\partial\sin\theta}$$

so $\tan\theta\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial\sin\theta}$ or

$$x\frac{\partial f}{\partial x} = lf \Rightarrow f \sim x^l \quad \text{thus:}$$

$x = \sin\theta \qquad Y_l^l(\theta,\varphi) \sim e^{il\varphi}\sin^l\theta \longleftarrow$ Still need normalization.

Normalizing:

$$\langle \hat{n} | l,l \rangle = Y_l^l(\theta,\varphi) = \frac{(-1)^l}{2^l\, l!}\sqrt{\frac{(2l+1)(2l)!}{4\pi}}\; e^{il\varphi}\sin^l\theta$$

phase factor chosen so that $Y_l^0(\theta,\varphi) = P_l(\cos\theta)$

Legendre polynomial

How do we get the rest of the Y_l^m's?

Ans. Apply the lowering operator L_-.

$$\langle \hat{n} | l,m-1 \rangle = \frac{\langle \hat{n} | L_- | l,m \rangle}{\hbar\sqrt{(l+m)(l-m+1)}} = \frac{e^{-i\varphi}}{\sqrt{(l+m)(l-m+1)}}\left(\frac{-\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\langle \hat{n} | l,m \rangle$$

$\Longrightarrow$

$$Y_l^m(\theta,\varphi) = \frac{(-1)^l}{2^l\, l!}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}}\;\frac{e^{im\varphi}}{\sin^m\theta}\;\frac{d^{\,l-m}}{d(\cos\theta)^{l-m}}(\sin\theta)^{2l}$$

$$Y_l^{-m}(\theta,\varphi) = (-1)^m\, Y_l^{m*}(\theta,\varphi)$$

Spherical Harmonics as rotation matrices:

$$|\hat{n}\rangle = \mathcal{D}(R)\,|\hat{z}\rangle = \mathcal{D}(\alpha=\varphi, \beta=\theta, \gamma=0)\,|\hat{z}\rangle$$

$$|\hat{n}\rangle = \sum_{l,m} \mathcal{D}(R)\,|l,m\rangle\langle l,m|\hat{z}\rangle$$

doesn't mess up l but mixes m's within one irreducible representation of fixed l.

$$\langle l,m'|\hat{n}\rangle = \sum_m \mathcal{D}^{(l)}_{m'm}(R)\,\langle l,m|\hat{z}\rangle$$

$Y_l^{m*}(\theta=0,\varphi)$ ← This vanishes for $m \neq 0$

undetermined

since $|\hat{z}\rangle$ is an eigenket of $L_z = x p_y - y p_x$

$$\Rightarrow \quad \langle l,m|\hat{z}\rangle = Y_l^{m*}(\theta=0,\varphi)\,\delta_{m0}$$

$$= \sqrt{\frac{(2l+1)}{4\pi}}\,P_l(\cos\theta)\Big|_{\cos\theta=1}\delta_{m0}$$

$$= \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m,0}$$

so

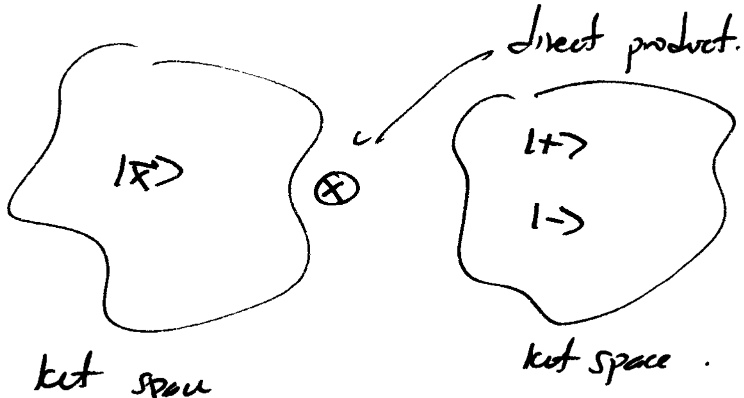
$$Y_l^{*m'}(\theta,\varphi) = \sqrt{\frac{(2l+1)}{4\pi}}\,\mathcal{D}^{(l)}_{m'0}(\alpha=\varphi,\beta=\theta,\gamma=0) \quad \text{or}$$

$$\mathcal{D}^{(l)}_{m0}(\alpha,\beta,\gamma=0) = \sqrt{\frac{4\pi}{(2l+1)}}\;Y_l^{m*}(\theta,\varphi)\Big|_{\theta=\beta,\,\alpha=\varphi}$$

Addition of Angular Momentum:

Spin-1/2 ket space and position eigenkets:

$$|\vec{x},\pm\rangle = |\vec{x}\rangle \otimes |\pm\rangle$$



The rotation operator $e^{-i\vec{J}\cdot\hat{n}\varphi/\hbar}$ is the same, but $\vec{J}$ must have <u>two</u> parts: $\vec{J} = \vec{L} + \vec{S}$

Really $\quad \vec{J} = \vec{L}\otimes \mathbb{1} + \mathbb{1}\otimes\vec{S}$

acts on $|x\rangle$ space

acts on $|\pm\rangle$ space.

3.29

The wavefunction $\langle \vec{x} \pm | \alpha \rangle$ is a spinor-valued function of $\vec{x}$

$$\langle \vec{x}, \pm | \alpha \rangle = \begin{pmatrix} \psi_+(\vec{x}) \\ \psi_-(\vec{x}) \end{pmatrix}$$

Another example: Two spin-½ objects.

$$\vec{S} = \vec{S}_1 + \vec{S}_2 = \vec{S}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}_2$$

$[S_{1x}, S_{2y}] = 0$ since they act in different ket-spaces.

op. eigenvalue.

Consider $S^2 = (\vec{S}_1 + \vec{S}_2)^2$: $s(s+1)\hbar^2$

$S_z = S_{1z} + S_{2z}$: $m\hbar$

S_{1z} : $m_1 \hbar$

S_{2z} : $m_2 \hbar$

We can use the eigenket of S^2 and S_z <u>or</u> S_{1z}, S_{2z} to form a basis for this ket space.

$\{m_1, m_2\}$ representation $|++\rangle$ $|+-\rangle$ $|-+\rangle$ $|--\rangle$