Chapter Three

3.1 Rotations

Note finite rotations generically do not commute.

Represent rotations by 3×3 real orthogonal matrices

$$R_{ij} V_j = V_i' \qquad \text{Now} \quad \vec{V}\cdot\vec{V} = \vec{V}'\cdot\vec{V}' \quad \text{so}$$

$$V_i'^{\,2} = R_{ij} V_j R_{ik} V_k = V_k (R^T_{ki} R_{ij}) V_j = V_k^2$$

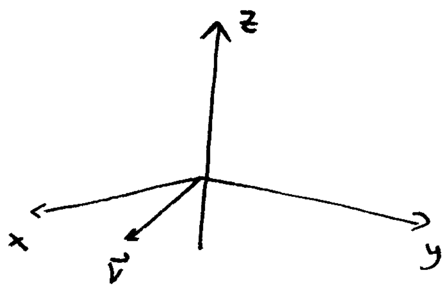
for all $\vec{V}$! $\Rightarrow$ It must be that $R^T_{ki} R_{ij} = R^T R = \mathbb{1}$

identity matrix

Thus $R^T R = R R^T = \mathbb{1}$ $\Leftarrow$ Rotation matrices are <u>orthogonal</u>

Consider a rotation about the z-axis [Active Rotations]

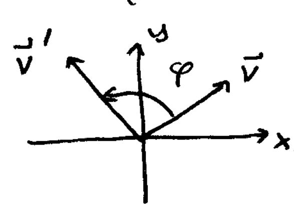
$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Consider

$$\overleftrightarrow{R}_z(\varphi)\cdot\vec{V} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} = \begin{pmatrix} V_x' \\ V_y' \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad V_x' = \cos\varphi\, V_x - \sin\varphi\, V_y$$
$$V_y' = \sin\varphi\, V_x + \cos V_y$$



ACTIVE ROTATION

Consider what happens when we let $\varphi \to \varepsilon \ll 1$

$$R_z(\varepsilon) = \begin{pmatrix} 1-\varepsilon^2/2 & -\varepsilon & 0 \\ \varepsilon & 1-\varepsilon^2/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_y(\varepsilon) = \begin{pmatrix} 1-\varepsilon^2/2 & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1-\varepsilon^2/2 \end{pmatrix}$$

$$R_x(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\varepsilon^2/2 & -\varepsilon \\ 0 & -\varepsilon & 1-\varepsilon^2/2 \end{pmatrix}$$

Note these are cyclic permutations of x, y, z

i.e. $x\ y\ z \rightarrow y\ z\ x \rightarrow z\ x\ y$

Now consider an infinitesimal rotation about y followed by one about x

$$R_x(\varepsilon) R_y(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\varepsilon^2/2 & -\varepsilon \\ 0 & \varepsilon & 1-\varepsilon^2/2 \end{pmatrix} \begin{pmatrix} 1-\varepsilon^2/2 & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1-\varepsilon^2/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\varepsilon^2/2 & 0 & \varepsilon \\ \varepsilon^2 & 1-\varepsilon^2/2 & -\varepsilon \\ -\varepsilon & \varepsilon & 1-\varepsilon^2/2 \end{pmatrix}$$

Now consider the reverse order of operations...

$$R_y(\varepsilon) R_x(\varepsilon) = \begin{pmatrix} 1-\varepsilon^2/2 & \varepsilon^2 & \varepsilon \\ 0 & 1-\varepsilon^2/2 & -\varepsilon \\ -\varepsilon & \varepsilon & 1-\varepsilon^2 \end{pmatrix}$$

dropping terms $\mathcal{O}(\varepsilon^2)$ and higher these operations (R_x, R_y) commute!

$$[R_x(\varepsilon), R_y(\varepsilon)] = \begin{pmatrix} 0 & -\varepsilon^2 & 0 \\ \varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R_z(\varepsilon^2) - \underset{R_{any}(0)}{1}$$

Infinitesimal Rotations in QM.

Given a rotation $R \leftarrow 3\times 3$ matrix there is an operator

$$|\alpha\rangle_R = \mathcal{D}(R)\,|\alpha\rangle$$

rotated ket — ket in original configuration

not a 3x3 matrix in general! This operator acts in the Hilbert space of state kets, not on vectors in Cartesian 3-space

To find $\mathcal{D}(R)$, let's consider an infinitesimal rotation. In analogy to time & space translations:

$$U_\varepsilon = 1 - iG\varepsilon \quad ; \text{ where } G \text{ is an Hermitian operator}$$

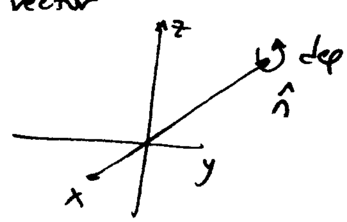
We will define the necessary Hermitian operator to be the angular momentum operator J_k it generates an infinitesimal rotation about the k^{th} axis by $d\varphi$ if we take

$$G\varepsilon \rightarrow \frac{J_k}{\hbar}d\varphi \quad \text{or}$$

unit vector

$$\mathcal{D}(\hat{n}, d\varphi) = 1 - \left(\frac{\vec{J}\cdot\hat{n}}{\hbar}\right)d\varphi$$

General Infinitesimal Rotation



why not $\vec{x}\times\vec{p}$? We'll come back to that!

We get a finite rotation in the usual way — by adding up lots of infinitesimal ones:

$$\mathcal{D}_z(\varphi) = \lim_{N\to\infty}\left[1 - i\frac{J_z\,\varphi}{\hbar\,N}\right]^N = e^{-iJ_z\varphi/\hbar}$$

We will show that ~~the representations~~ of the rotation group $\mathcal{D}(R)$ themselves form a group. We will consider these ~~f~~ more ~~formal~~ issues later.

The behavior of the rotation group near the identity:

Consider

$$\mathcal{D}(R_x^{(\varepsilon)})\,\mathcal{D}(R_y^{(\varepsilon)}) - \mathcal{D}(R_y^{(\varepsilon)})\,\mathcal{D}(R_x^{(\varepsilon)}) \quad \text{where all } R\text{'s are near } 1$$
$$= R_z(\varepsilon^2) - 1$$

$$= \left(1 - \frac{i J_x \varepsilon}{\hbar} - \frac{J_x^2 \varepsilon^2}{2\hbar^2}\right)\left(1 - \frac{i J_y \varepsilon}{\hbar} - \frac{J_y^2 \varepsilon^2}{\hbar^2}\right) - \left(1 - \frac{i J_y \varepsilon}{\hbar} - \frac{J_y^2 \varepsilon^2}{\hbar^2}\right) \cdot$$

$$\cdot \left(1 - \frac{i J_x \varepsilon}{\hbar} - \frac{J_x^2 \varepsilon^2}{\hbar^2}\right) = 1 - \frac{i J_z}{\hbar^2}\varepsilon^2 - 1$$

$$\Rightarrow \quad J_x J_y - J_y J_x = i\hbar J_z \quad \text{or} \quad \boxed{[J_x, J_y] = i\hbar J_z}$$

and symmetrically for cyclic permutations of (x, y, z).

Fundamental Commutation Relations of Angular Momentum

Note: The generators (elements of the Lie algebra) do not commute $\Leftrightarrow$ Non-abelian group.

Spin-1/2 and Finite Rotations:

What is the smallest dimension N for which we can have non-commuting operators?

$N = 0$? Don't know what that means.
$N = 1$ ✗ number multiplications commute.
$N \geq 2$ ← Can have non-commuting matrices!

$\Rightarrow$ Smallest representation for the angular momentum algebra can be found for 2×2 matrices $\rightarrow$ we already had a name for that:

Spin-1/2

Instead of $\vec{J}$ we called these $\vec{S}$:

$$S_x = \frac{\hbar}{2}\left\{ |+\rangle\langle -| + |-\rangle\langle +| \right\}$$

$$S_y = \frac{i\hbar}{2}\left\{ -|+\rangle\langle -| + |-\rangle\langle +| \right\}$$

$$S_z\ \frac{\hbar}{2}\left\{ |+\rangle\langle +| - |-\rangle\langle -| \right\}$$ ← In the S_z eigenket basis.

Consider a finite Rotation by φ about the $\hat{z}$ axis.

$$\mathcal{D}_z(\varphi) = e^{-iS_z\varphi/\hbar}$$

What does this do to an expectation value of S_z?

$$\langle S_z\rangle = \langle\alpha|S_z|\alpha\rangle \longrightarrow {}_R\langle\alpha|S_z|\alpha\rangle_R$$

$$\longrightarrow \langle\alpha| \underbrace{e^{+iS_z\varphi/\hbar} S_z}_{\text{commute}} e^{-iS_z\varphi/\hbar}|\alpha\rangle = 0.$$

What about $\langle S_x\rangle$

$$\langle S_x\rangle \longrightarrow \langle\alpha|\underbrace{e^{iS_z\varphi/\hbar} S_x e^{-iS_z\varphi/\hbar}}_{\text{work this out...}}|\alpha\rangle$$

Version One:

$$e^{iS_z\varphi/\hbar} S_x e^{-iS_z\varphi/\hbar} = e^{iS_z\varphi/\hbar}\frac{\hbar}{2}\left\{ |+\rangle\langle -| + |-\rangle\langle +| \right\} e^{-iS_z\varphi/\hbar}$$

$$= \frac{\hbar}{2}\left\{ e^{+i\varphi/2}|+\rangle\langle -|e^{+i\varphi/2} + e^{-i\varphi/2}|-\rangle\langle +|e^{-i\varphi/2} \right\}$$

$$= \frac{\hbar}{2}\left\{ \left(|+\rangle\langle -| + |-\rangle\langle +|\right)\cos\varphi + i\sin\varphi\left(|+\rangle\langle -| - |-\rangle\langle +|\right) \right\}$$

$$= \cos\varphi\, S_x - \sin\varphi\, S_y$$

Version Two:

Use the Baker-Hausdorff lemma:

$$e^{iS_z\varphi/\hbar}\, S_x\, e^{-iS_z\varphi/\hbar} = S_x + \frac{i\varphi}{\hbar}\underbrace{[S_z, S_x]}_{i\hbar S_y} + \frac{(i\varphi)^2}{2\hbar^2}\underbrace{[S_z\, \underbrace{[S_z, S_x]}_{i\hbar S_y}]}_{\hbar^2 S_x}$$

$$+ \frac{1}{3!}\left(\frac{i\varphi}{\hbar}\right)^3 \underbrace{[S_z, \underbrace{[S_z, [S_z, S_x]]}_{\hbar^2 S_x}]}_{i\hbar^3 S_y} + \cdots$$

Even terms all have S_x Odd terms all have S_y

$$\Rightarrow e^{iS_z\varphi/\hbar}\, S_x\, e^{-iS_z\varphi/\hbar} = S_x\left[1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots\right]$$

$$- S_y\left[\varphi - \frac{\varphi^3}{3!} + \cdots\right]$$

$= S_x \cos\varphi - S_y \sin\varphi$ as expected!

So: what have we learned

$$\langle S_x\rangle \longrightarrow {}_R\langle\alpha|S_x|\alpha\rangle_R = \langle S_x\rangle\cos\varphi - \langle S_y\rangle\sin\varphi$$

$$\langle S_y\rangle \longrightarrow {}_R\langle\alpha|S_y|\alpha\rangle_R = \langle S_y\rangle\cos\varphi + \langle S_x\rangle\sin\varphi$$

$$\langle S_z\rangle \rightarrow \langle S_z\rangle$$

so $$\langle S_z\rangle \longrightarrow \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \langle S_x\rangle \\ \langle S_y\rangle \\ \langle S_z\rangle \end{pmatrix}$$

or

$\langle S_k\rangle = \mathcal{R}_{k\ell}\langle S_\ell\rangle$ — the total spin expectation value rotates like an ordinary vector

Later we will generalize this for vector and tensor operators

Now for a new feature:

Consider a general ket: $|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle$

$$e^{-iS_z\varphi/\hbar}|\alpha\rangle = e^{-i\varphi/2}|+\rangle\langle+|\alpha\rangle + e^{i\varphi/2}|-\rangle\langle-|\alpha\rangle$$

the appearance of this ½ angle has one important implication

$$|\alpha\rangle_{R_z(2\pi)} = -|\alpha\rangle \Longleftarrow$$ You don't get the same ket back again!

rotate the ket by 2π

Question: Is this minus sign observable?

Neutron interference and the sign under a 2π- rotation of a ket.

Spin Precession $H = -\dfrac{e}{m_e c}\vec{S}\cdot\vec{B} = \omega S_z$; $\omega = \dfrac{|e|B}{m_e c}$

So the time evolution operator is simply related to rotations

$$\mathcal{U}(t,0) = e^{-iHt/\hbar} = e^{-i\omega t S_z/\hbar}$$

From our previous results we have:

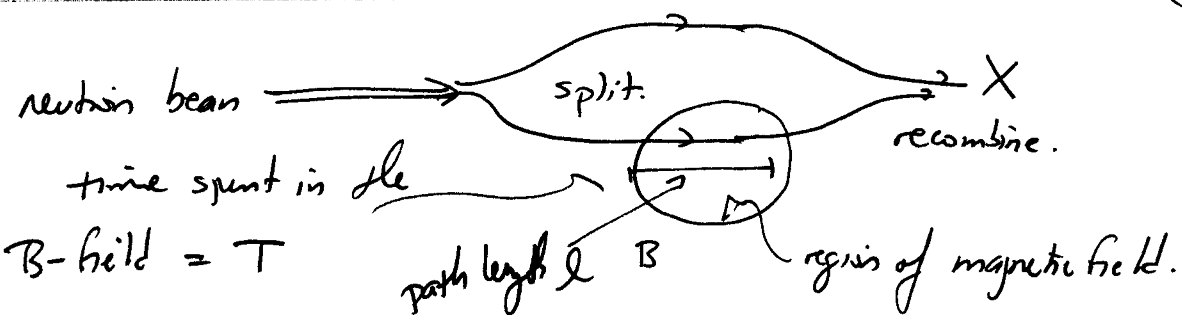
$$\langle S_x\rangle(t) = \langle S_x\rangle(0)\cos(\omega t) - \langle S_y\rangle(0)\sin\omega t$$

$$\langle S_y\rangle(t) = \langle S_x\rangle(0)\sin(\omega t) + \langle S_y\rangle(0)\cos\omega t$$

$$\langle S_z\rangle(t) = \langle S_z\rangle(0)$$

We can use these equations to describe the precession of neutrons as well if we change $\omega = \dfrac{g_n e B}{m_p c}$ $\quad g_n \cong -1.91$

Consider the following experiment:



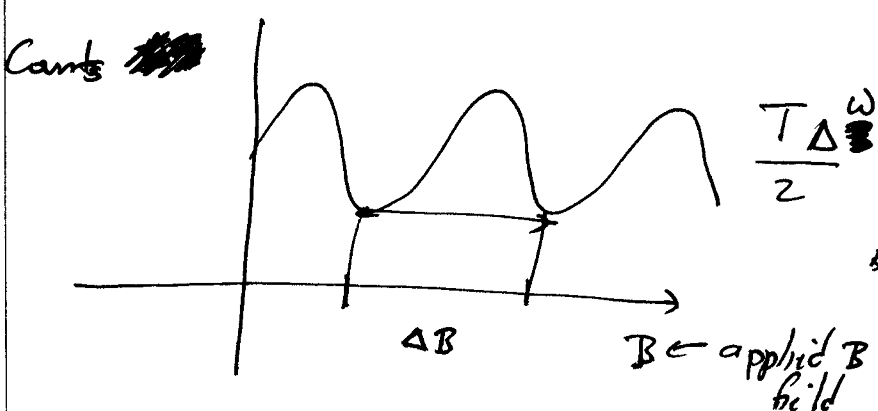
consider a ket arriving at X by path A $|\alpha\rangle$

" " by path B $|\alpha\rangle e^{i\varphi} e^{i\omega T/2}$

Prob of finding a neutron $\propto$ $\left|\langle\alpha| + e^{-i\varphi} e^{-i\omega T/2}\langle\alpha| \cdot |\alpha\rangle + e^{i\varphi} e^{i\omega T/2}|\alpha\rangle\right|^2$

path A path B

$= |\langle\alpha|\alpha\rangle|^2 + |\langle\alpha|\alpha\rangle|^2\, 2\cos(\varphi + \omega T/2)$. Now can change the value of B so the phase interference cycles around.



$$\frac{T\Delta\omega}{2} = \frac{g_n e \Delta B}{2 m_p c} T = 2\pi$$

$$\Rightarrow \Delta B = 4\pi \frac{T m_p c}{g_n e}$$

~~can write~~ $\cancel{T = \frac{l}{v} = \frac{l}{m_p p}}$

Need $\frac{\omega T}{2} \rightarrow \frac{(\omega + \Delta\omega)T}{2} = \frac{\omega T}{2} + 2\pi$

$$\Rightarrow \frac{\Delta\omega T}{2} = 2\pi \Rightarrow \Delta\omega = \frac{4\pi}{T} \Rightarrow \frac{g_n e \Delta B}{m_p c} = \frac{4\pi}{T}$$

$$\Rightarrow \Delta B = \frac{4\pi}{T}\frac{m_p c}{g_n e} \ ; \quad T = \frac{l}{v} = \frac{l m_p}{p} =$$

$p = \hbar k = \frac{2\pi}{\lambda}\hbar$

$$\Delta B = \frac{4\pi m_p c}{g_n e} \times \left(\frac{\ell m_p}{\hbar/ƛ}\right)^{-1} = \frac{4\pi \hbar c}{g_n e ƛ \ell}$$

See S. A. Werner et al. PRL $\underline{35}$, 1053 (1975)

$\frac{\lambda}{2\pi}$ — wavelength of the cold neutrons.

Here we have an explicit result showing that nature uses the spin-1/2 Representation of the rotation group!

Note: there can be no classical analogue. If you turn your pencil around by 2π, it is still your pencil, not minus your pencil!

Pauli Two-Component Formalism for spin-1/2

We already know this.

$$|+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} \equiv \chi_+ \quad \text{and} \quad \langle +| \doteq (1, 0) = \chi_+^{\dagger}$$

$$|-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix} \equiv \chi_-$$

An arbitrary two-component spinor is: $\chi = \begin{pmatrix}\langle +|\alpha\rangle\\ \langle -|\alpha\rangle\end{pmatrix} = \begin{pmatrix}c_+\\ c_-\end{pmatrix}$

$$\chi = c_+ \chi_+ + c_- \chi_-$$

spinors.

c-numbers.

The matrix elements of $\langle \pm | S_k | \pm \rangle$ are $\frac{\hbar}{2} \times$ a Pauli matrix.

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Note: $\sigma_i^2 = 1$ (no sum)

$\sigma_i \sigma_j + \sigma_j \sigma_i = 1$ for $i \neq j$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$ ← They obey the angular moment commutation relation

Also $\mathrm{Det}(\sigma_i) = -1$

$\sigma_i^\dagger = \sigma_i$ and $\mathrm{Tr}(\sigma_i) = 0$

For any vector $\vec{a}$: $\vec{\sigma}\cdot\vec{a} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$ has the properties:

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

in particular: $(\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2$.

Rotations in the Pauli Formalism:

$$\mathcal{D}(\hat{n},\varphi) = e^{i\varphi\vec{S}\cdot\hat{n}/\hbar} \doteq e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2}$$ ← using $(\vec{\sigma}\cdot\hat{n})^{2n} = 1$

$(\vec{\sigma}\cdot\hat{n})^{2n+1} = \vec{\sigma}\cdot\hat{n}$

$$e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2} = \mathbb{1}\cos(\varphi/2) - i\vec{\sigma}\cdot\hat{n}\sin(\varphi/2)$$

So $e^{-i\vec{S}\cdot\hat{n}\varphi/\hbar}|\alpha\rangle \doteq e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2}\chi$ so that under rotations

$$\chi \to e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2}\chi$$

Note: $\vec{\sigma}$ does not change under rotation $\Rightarrow$ $\vec{\sigma}$ is not a vector but

$$\chi^\dagger\sigma_k\chi \to \sum_l R_{kl}\,\chi^\dagger\sigma_l\chi$$ transforms as a vector.

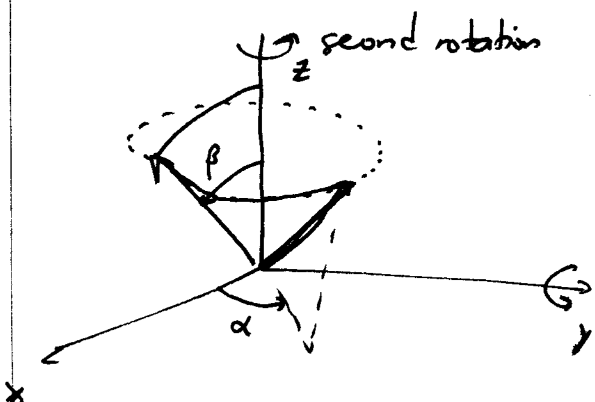
You can check this from:

$$e^{i\sigma_3\varphi/2}\sigma_1 e^{-i\sigma_3\varphi/2} = \sigma_1\cos\varphi - \sigma_2\sin\varphi$$ and so on.

What are the eigenspinors of $\vec{\sigma}\cdot\hat{n}$?

What we want is $|\vec{S}\cdot\hat{n};+\rangle$ so that

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \frac{\hbar}{2}|\vec{S}\cdot\hat{n};+\rangle$$



start w/ the $|S_z;+\rangle$ spinor $\begin{pmatrix}1\\0\end{pmatrix}$ and just rotate it to $\hat{n}$ for an arbitary $\hat{n}$ specified by the polar angles β, α.

$$e^{-i\sigma_2\beta/2}\begin{pmatrix}1\\0\end{pmatrix} = \left\{\cos(\beta/2) - i\,\sigma_z \sin(\beta/2)\right\}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and}$$

$$e^{-i\sigma_3\alpha/2}e^{-i\sigma_2\beta/2}\begin{pmatrix}1\\0\end{pmatrix} = \left\{\cos(\alpha/2) - i\,\sigma_3 \sin(\alpha/2)\right\}\left\{\cos(\beta/2) - i\,\sigma_2 \sin(\beta/2)\right\}\begin{pmatrix}1\\0\end{pmatrix}$$

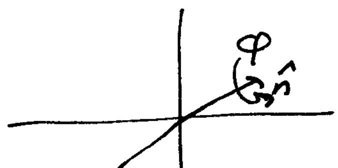
$$= \begin{pmatrix}\cos(\alpha/2) - i\sin(\alpha/2) & 0\\ 0 & \cos(\alpha/2) - i\sin(\alpha/2)\end{pmatrix}\begin{pmatrix}\cos(\beta/2) & -\sin(\beta/2)\\ \sin(\beta/2) & \cos(\beta/2)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\beta/2)\,e^{-i\alpha/2}\\ \sin(\beta/2)\,e^{i\alpha/2}\end{pmatrix}$$

O(3), SU(2), and Euler Angles.

A beginning to the study of groups in Q.M.

We have characterized rotations by orthogonal matrices. R.
How many numbers are involved in defining a rotation?
Ans 3



$\hat{n} = 2\#$ (independent)
$\varphi = 1\#$.

What about R? $\Leftarrow$ 3x3 matrix (real) = 9 #s.
But orthogonality $RR^T = \mathbb{1}$ $\Leftarrow$ 6 equations!
why? $\quad RR^T = \mathbb{1} \Leftarrow$ 3 equations.
$\quad R^T R = \mathbb{1} \Leftarrow$ 3 more.
$\Rightarrow$ 9-6 = 3 independent #s - good!
Check that R's form a group under matrix multiplication.

1. $R_1, R_2 \in O(3) \Rightarrow R_1 R_2 \in O(3)$?
$(R_1 R_2)(R_1 R_2)^T = R_1 R_2 R_2^T R_1^T = R_1 R_1^T = \mathbb{1}$. yes.
2. Clearly associative.
3. Identity $\mathbb{1}$.
4. Inverse is R^{-1} a rotation in the other direction.

We also saw we can characterize a rotation of a spinor by a 2x2 matrix $e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2}$ acting on a spinor χ.

This matrix is unitary: $\left(e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2}\right)^\dagger = e^{i\vec{\sigma}\cdot\hat{n}\varphi/2} = \left(e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2}\right)^{-1}$

It is also unimodular $\Rightarrow$ has determinant 1.

Most general unitary, unimodular matrix U:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad a, b = \text{Cayley-Klein parameters.}$$

Note: $a = \cos(\varphi/2) - n_z i \sin(\varphi/2)$
$b = -n_y \sin(\varphi/2) - n_x \sin(\varphi/2)$.

$\det U = |a|^2 + |b|^2 = 1 \leftarrow$ condition on the complex numbers a and b.

Note $U^+ = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$ so

$$U U^+ = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} |a|^2+|b|^2 & -ab+ab \\ -a^*b^*+a^*b^* & |b|^2+|a|^2 \end{pmatrix}$$

$= \mathbb{1}$. check!

Note: $a, b \Rightarrow 4$ #s

$|a|^2 + |b|^2 = 1 \leftarrow 1$ equation

Again: 3 independent numbers to specify a rotation!

The group of U's is called $SU(2)$

special i.e. unimodular, unitary, 2x2

The most general form is:

$$U = e^{i\gamma}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad |a|^2 + |b|^2 = 1, \; \gamma = \gamma^*$$

4 parameters.

We can characterize rotations by either $O(3)$ or $SU(2)$. Are they the same group? (i.e. isomorphic?)

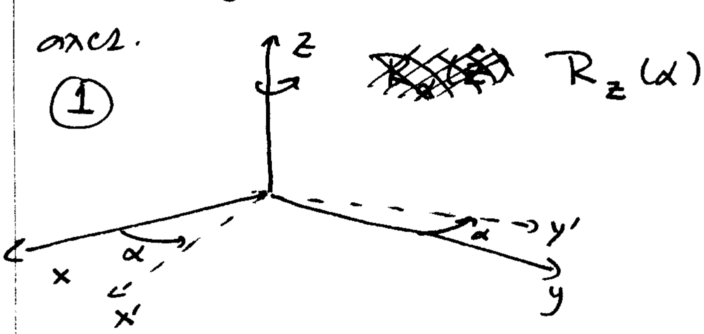
Ans. No, only locally, but not globally.

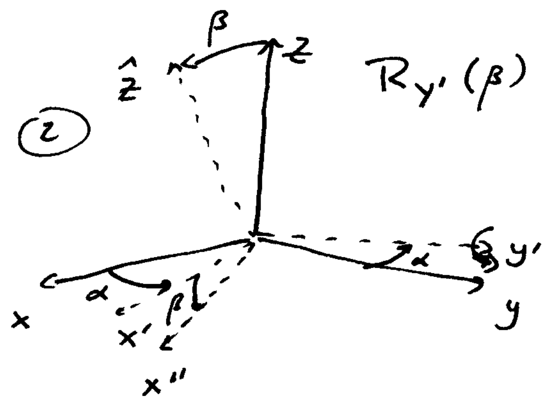
A rotation by 2π and one by 4π are both $\mathbb{1} \in O(3)$

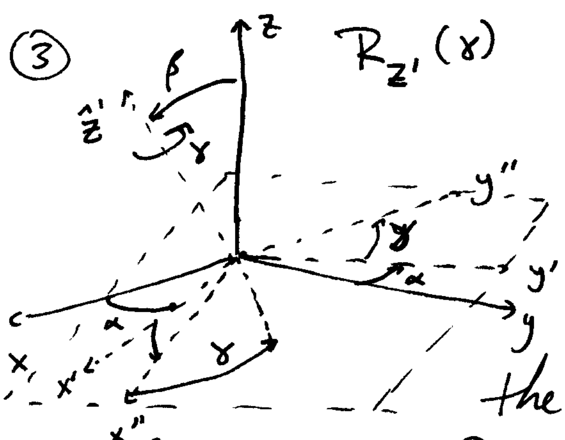
In $SU(2)$ these correspond to $-\mathbb{1}$ and $\mathbb{1}$ respectively.

In general $U(a,b)$ and $U(-a;-b)$ correspond to the same 3×3 matrix in $O(3)$. $\Rightarrow$ There is a 2 to 1 mapping of $SU(2)$ to $O(3)$.

It is useful to characterise an arbitrary rotation in terms of <u>Euler Angles</u>: Three rotations about ~~space-fixed~~ specified axes.







$$\Rightarrow \quad R = \underbrace{R_{z'}(\gamma)\, R_{y'}(\beta)}_{\text{body axes}}\, \underset{\substack{\uparrow \\ \text{space axis}}}{R_z(\alpha)}$$

We know how to rotate wrt the space-fixed axes but not the body-fixed ones. What to do?

Well, $$R_{y'}(\beta) = R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha)$$

Similarly $$R_{z'}(\gamma) = R_{y'}(\beta)\, R_z(\gamma)\, R_{y'}^{-1}(\beta)$$

$$\Rightarrow R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)$$ $\leftarrow$ Now, all space-fixed!

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Apply this to the spin-1/2 system:

$$\mathcal{D}(\alpha,\beta,\gamma) = \mathcal{D}_z(\alpha)\,\mathcal{D}_y(\beta)\,\mathcal{D}_z(\gamma)$$

$$= e^{-i\sigma_3\alpha/2}\, e^{-i\sigma_2\beta/2}\, e^{-i\sigma_3\gamma/2}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}$$

$\mathcal{D}^{(1/2)}_{m'm}(\alpha,\beta,\gamma)$ $j = 1/2$ irreducible representation of the rotation operator.

More representations:

$[J^2, J_k] = 0 \quad k = x, y, z$

$J^2 |a,b\rangle = a|a,b\rangle$

$J_z |a,b\rangle = b|a,b\rangle$

ladder ops. $J_\pm = J_x \pm i J_y$

$[J_z, J_\pm] = \pm\hbar J_\pm$

$[J_+, J_-] = 2\hbar J_z$

$[J^2, J_\pm] = 0$

$$\Rightarrow \quad J_z J_\pm |a,b\rangle = (b \pm \hbar) J_\pm |a,b\rangle$$

and

$$J^2 J_\pm |a,b\rangle = a\, J_\pm |a,b\rangle$$

Prove $a \geq b^2$

$$J^2 - J_z^2 = \tfrac{1}{2}(J_+J_- + J_-J_+) = \tfrac{1}{2}(J_+J_+^\dagger + J_+^\dagger J_+)$$

$$\Rightarrow \quad \langle a,b|J^2 - J_z^2|a,b\rangle \geq 0$$

has nonnegative expectation value: $\langle\alpha|A^\dagger A|\alpha\rangle \geq 0$

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$\exists\, b_{max}$ $\quad J_+ |a, b_{max}\rangle = 0$ $\qquad J_- J_+ = J_x^2 + J_y^2 - i(J_y J_x - J_x J_y)$

$$J_- J_+ |a, b_{max}\rangle = (J^2 - J_z^2 - \hbar J_z)\,|a, b_{max}\rangle = 0$$

$$\Rightarrow a - b_{max}^2 - b_{max}\hbar = 0 \Rightarrow a = b_{max}(b_{max} + \hbar)$$

Similarly $\quad J_- |a, b_{min}\rangle = 0 \qquad J_+ J_- = J^2 - J_z^2 + \hbar J_z$

$$\Rightarrow a = b_{min}(b_{min} - \hbar)$$

$$\Rightarrow b_{max} = -b_{min} \Rightarrow -b_{max} \le b \le b_{max}$$

$b_{max} = b_{min} + n\hbar$

$\Rightarrow b_{max} = n\hbar/2$ $\quad \leftarrow$ normally call this j.

$j = n/2 \;\rightarrow\; a = \hbar^2 j(j+1)$

$b = m\hbar \qquad m = -j, \ldots, j$