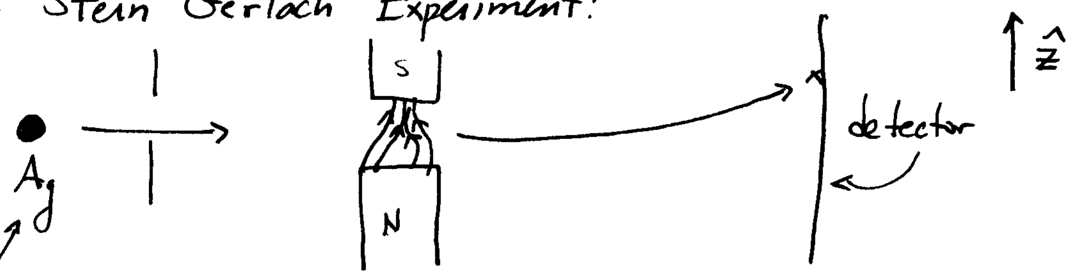
Fundamental Ideas in Q.M.

The Stern Gerlach Experiment:



$\vec{\mu} \propto \vec{S}$

atomic magnetic moment

one electron spin

actually $\vec{\mu} = \frac{e}{m_e c}\vec{S}$ to an accuracy of about 0.2%.

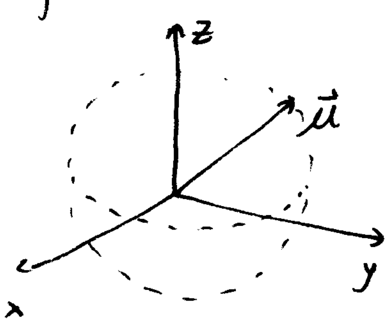
Force on the atom: $F_z = \frac{\partial}{\partial z}(\vec{\mu}\cdot\vec{B}) \simeq \mu_z \frac{\partial B_z}{\partial z}$

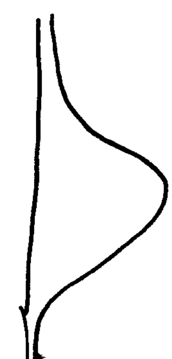
proportional to the magnitude of the atoms z-component of its magnetic moment $\vec{\mu}$.

$\mu_z > 0$ i.e. $S_z < 0$ (Recall $e<0$) downward force

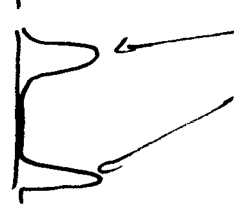
$\mu_z < 0$ i.e. $S_z > 0$ upward force.

For a randomly oriented set of spins of the Ag atoms we expect at the detector:





Instead:

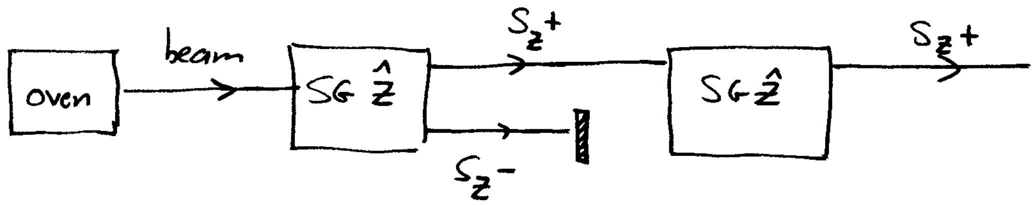


Two possible values of $S_z = \pm\hbar/2$

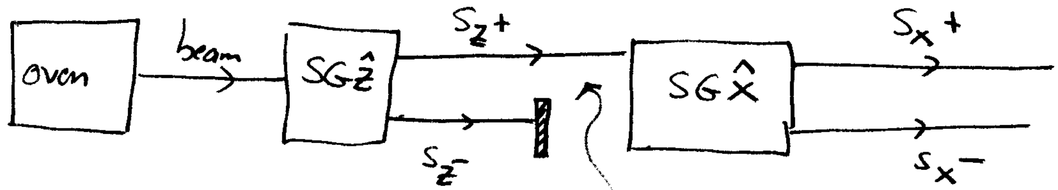
Where $\hbar = 1.05 \times 10^{-27}\ \text{erg}\cdot\text{s} = 6.58 \times 10^{-16}\ \text{eV}\cdot\text{s}$

fundamental unit of angular momentum.

Multiple SG experiments: Analogy to polaroid filters.

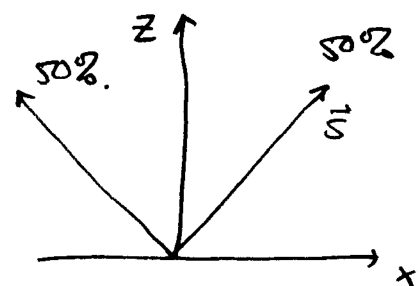


No S_z- beam at all!

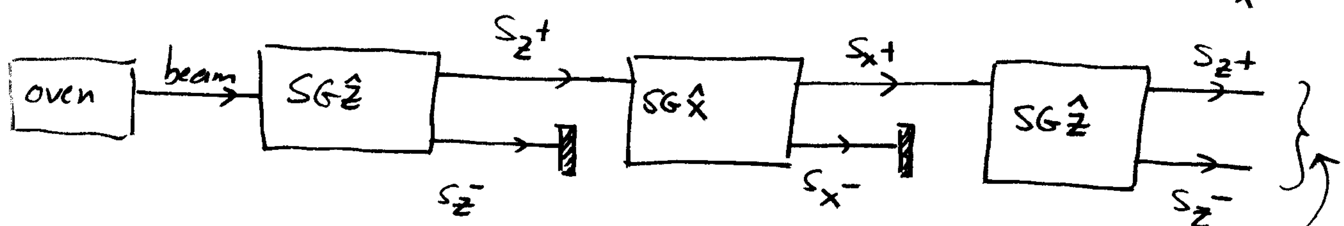


} equal intensities

Can we say that here 50% of the atoms have S_x+ and 50% have S_x- while all have S_z+



NO.



We get both beams again!

The SG $\hat{x}$ machine destroys information about S_z!

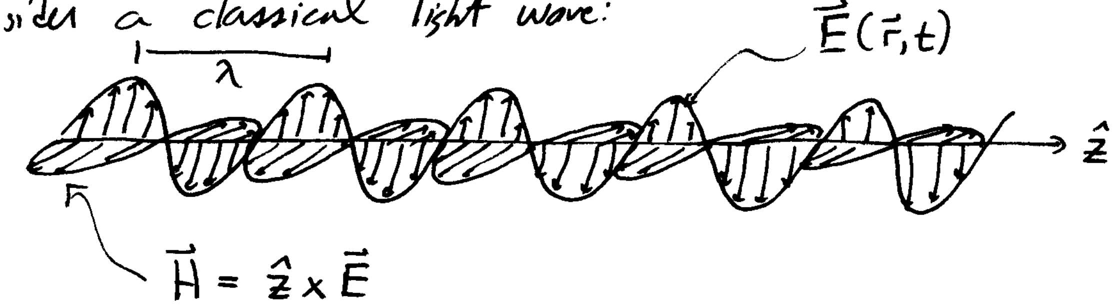
This is completely different from the classical notion of angular momentum. We can measure all three components of $\vec{L} = \vec{r} \times \vec{p}$ in any order and completely define the vector $\vec{L}$.

Analogy to the Polarization of Light:

Classical Physics + Quantization of Light $\Rightarrow$ probabilistic interpretation in quantum mechanics.

Quick Review of Classical E+M

Consider a classical light wave:



$$\vec{E}(\vec{r},t) = \begin{pmatrix} E_x(\vec{r},t) \\ E_y(\vec{r},t) \\ 0 \end{pmatrix} \quad ; \quad E_{x,y}(\vec{r},t) = E^0_{x,y} \cos(kz - \omega t + \alpha_{x,y})$$

$k = \frac{2\pi}{\lambda}$ wavenumber $\quad \omega$ = angular frequency

$\alpha_{x,y}$ phases and $E^0_{x,y}$ amplitudes

From now on it will be easier to use complex notation:

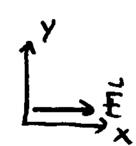
$$E_x' = E_x^0 e^{i\alpha_x}, \quad E_y' = E_y^0 e^{i\alpha_y} \quad \text{and}$$

$$E_x = E_x' e^{i(kz-\omega t)} \quad E_y = E_y' e^{i(kz-\omega t)}$$

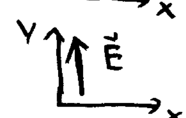
Recall that to get the physical field we just take the real parts.

The Polarization state is controlled by E_x', E_y':

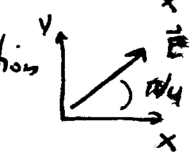
$E_y' = 0$ plane polarized in the x direction



$E_x' = 0$ plane polarized in the y direction



$E_x' = E_y'$ plane polarized in the $\frac{\hat{x}+\hat{y}}{\sqrt{2}}$ direction



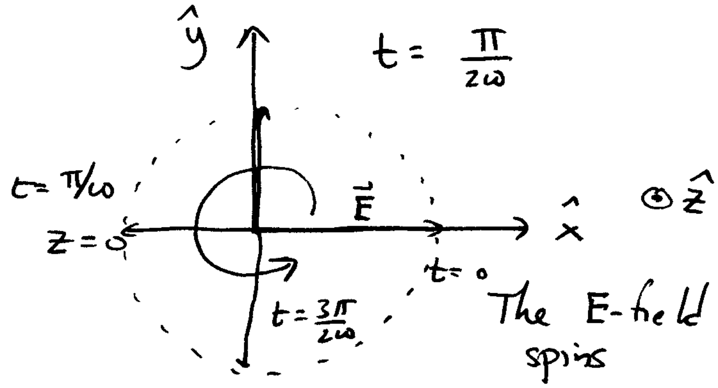
$$E_y' = i E_x' = e^{i\pi/2} E_x' \quad \underline{\underline{\text{Right}}} \text{ circularly polarized}$$

$$\text{Re}\, E_y \sim \cos(kz - \omega t + \pi/2)$$

$$\text{Re}\, E_x \sim \cos(kz - \omega t)$$

$$\text{Re}\, E_y\Big|_{z=0} \sim \cos(\omega t - \pi/2)$$

$$\text{Re}\, E_x\Big|_{z=0} \sim \cos(\omega t)$$



$$E_y' = -i E_x' = e^{-i\pi/2} E_x' \quad \underline{\underline{\text{Left}}} \text{ circularly polarized}$$

The energy density of the electromagnetic field (Gaussian units) is

$$\mathcal{E}(\vec{r},t) = \frac{1}{8\pi}\left[|\text{Re}\, \vec{E}|^2 + |\text{Re}\, \vec{H}|^2 \right] ; \quad \text{we have } \text{Re}\, H_x = -\text{Re}\, E_y, \quad \text{Re}\, H_y = \text{Re}\, E_x$$

so

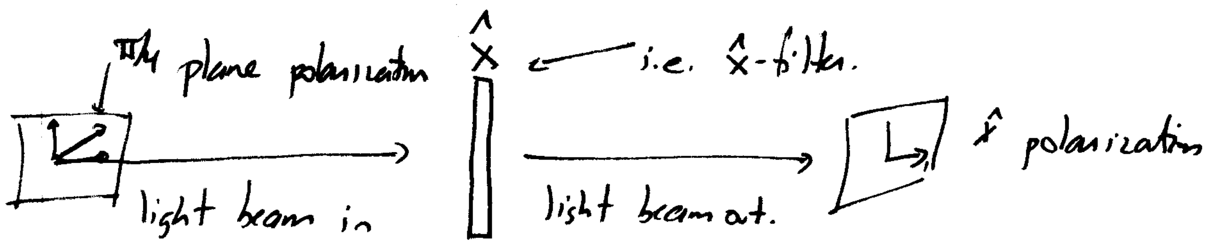
$$\mathcal{E}(\vec{r},t) = \frac{1}{4\pi}\left[(\text{Re}\, E_x)^2 + (\text{Re}\, E_y)^2 \right]$$

$$= \frac{1}{4\pi}\left[|E_x|^2 \cos(kz - \omega t + \alpha_x) + |E_y|^2 \cos(kz - \omega t + \alpha_y) \right]$$

If the wave occupies a volume V where V is many λ in the $\hat{z}$ direction

$$\mathcal{E}_{total} = \int_V d^3\vec{r}\, \mathcal{E}(\vec{r},t) = \frac{V}{4\pi} \frac{1}{2} \left(|E_x|^2 + |E_y|^2 \right) = \frac{|\vec{E}|^2 V}{8\pi}$$

$$\Rightarrow \frac{|\vec{E}|^2}{8\pi} = \text{average energy per unit volume.}$$



$E_x = E_y = E$ polaroid filter $E_x = E$ $E_y = \underline{\underline{0}}$

no E-field in $\hat{y}$.

How can we think about this knowing that the total energy of the beam is <u>quantized</u>: $\mathcal{E}_{total} = N\hbar\omega$

number of photons.

<u>Incoming</u>: <u>Outgoing</u>:

$$\frac{\mathcal{E}_{total}}{V} = \frac{2E^2}{8\pi} = \frac{E^2}{4\pi} \quad ; \quad \frac{\mathcal{E}_{total}}{V} = \frac{E^2}{8\pi}.$$

$\Rightarrow \quad N_{incoming} = 2\,N_{outgoing}$ Half of the photons don't get through.

All photons are the same $\Rightarrow$ each has a probability of 1/2 of getting through.

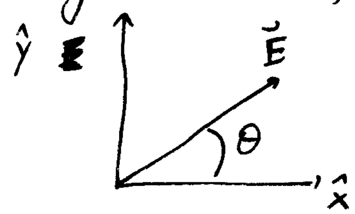
Mean numbers give the classical result, but fluctuations about this are purely quantum mechanical. In the limit that $N \to \infty$ we recover the classical result. — Correspondence Principle.

We now have a way to calculate the probability of a photon going through the polarizer.

Say we have an incoming beam with E_x, E_y going through a $\hat{x}$ filter

$\Rightarrow$ Each photon has a probability of $\dfrac{|E_x|^2}{|E_x|^2+|E_y|^2}$ of getting through:

e.g. beam plane polarized at angle angle θ



hitting an x-filter:

prob of passing $\dfrac{|E_x|^2}{|E|^2} = \cos^2\theta$ ← all photons that pass are polarized in the $\hat{x}$-direction!

☆ There is nothing special about the $\{\hat{x}, \hat{y}\}$ basis:
We could have told the above story using Right and left circularly polarized photons. An RCP filter passes only Right circularly polarized light.

If the incoming beam is $\vec{E} = \vec{E}_{RCP} + \vec{E}_{LCP}$ the probability that a photon goes through is $\dfrac{|E_{RCP}|^2}{|E|^2}$

Defining a polarization state vector.

For one photon $\dfrac{|\vec{E}|^2 V}{8\pi} = \hbar\omega$ ← $(N=1)$

define $|\psi\rangle = \begin{pmatrix}\psi_x \\ \psi_y\end{pmatrix}$; $\psi_{x,y} = \sqrt{\dfrac{V}{8\pi\hbar\omega}}\, E_{x,y}$

$|\psi\rangle$ are vectors in a complex 2-d space.

We have normalized $|\psi\rangle$ so that

$$|\psi_x|^2 + |\psi_y|^2 = 1.$$

The polarization states we looked at earlier can be written as:

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{x}\text{- polarized plane wave.}$$

$$|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \hat{y}\text{- " "}$$

$$|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \text{RCP}$$

$$|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \text{LCP}$$

We define dual vectors as rows

$$\langle\psi| = (\psi_x^*, \psi_y^*)$$

complex conjugate

and an inner product $\langle\varphi|\psi\rangle = \varphi_x^* \psi_x + \varphi_y^* \psi_y$

$$= \langle\psi|\varphi\rangle^*$$

Note the normalization condition can be written as: $\langle\psi|\psi\rangle = 1$

Note also that $\langle x|y\rangle = 0$; $|x\rangle, |y\rangle$ are orthogonal.

also: $\langle R|R\rangle = \frac{1}{\sqrt{2}}(1 \; -i)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(1 \; -i)\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\cdot 2 = 1$

Similarly $\langle L|L\rangle = 1$ and

$$\langle L|R\rangle = \frac{1}{2}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(1-1) = 0$$

We can take either $\{|x\rangle, |y\rangle\}$ or $\{|R\rangle, |L\rangle\}$ as an orthonormal basis

For any polarization state: $|\psi\rangle$

$$|\psi\rangle = \psi_x |x\rangle + \psi_y |y\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

but also

$$|\psi\rangle = |R\rangle\langle R|\psi\rangle + |L\rangle\langle L|\psi\rangle$$

what are $\langle R|\psi\rangle$, $\langle L|\psi\rangle$ in terms of ψ_x, ψ_y?

$$\langle R|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \frac{1}{\sqrt{2}}(\psi_x - i\psi_y)$$

$$\langle L|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \frac{1}{\sqrt{2}}(\psi_x + i\psi_y)$$

Superposition Principle: We can think of $|\psi\rangle$ as a <u>coherent</u> superposition of $\{|x\rangle\ |y\rangle\}$ or $\{|R\rangle, |L\rangle\}$.

explained later

Note we can write the above as follows:

$$|\psi\rangle = \big(|x\rangle\langle x| + |y\rangle\langle y|\big)\cdot|\psi\rangle \quad \text{for all } |\psi\rangle$$

$$= \big(|R\rangle\langle R| + |L\rangle\langle L|\big)\cdot|\psi\rangle \quad \text{"} \quad \text{"}$$

Or $\quad |x\rangle\langle x| + |y\rangle\langle y| = \mathbb{1} \leftarrow$ Identity

$\quad |R\rangle\langle R| + |L\rangle\langle L| = \mathbb{1}$

This is an example of closure or the completeness relation.

Consider passing the beam through the $\hat{x}$-filter

The fraction of energy of the beam that passes through is

$\dfrac{|E_x|^2}{|E|^2} \propto$ probability of an individual photon passing through.

$$\text{Prob} = \frac{|\psi_x|^2}{|\psi_x|^2 + |\psi_y|^2} = |\psi_x|^2 = |\langle x|\psi\rangle|^2$$

$\langle x|\psi\rangle =$ amplitude of the $\hat{x}$ polarized component of $|\psi\rangle$

Probability amplitude. It's absolute square is the probability.

In general if we have a measurement device (polarizers or SG device) that passes particles in state $|\varphi\rangle$ and rejects those in states orthogonal to $|\varphi\rangle$, then the prob amplitude that a particle passes in state $|\psi\rangle$ is

$\langle\varphi|\psi\rangle$ and the probability is

$|\langle\varphi|\psi\rangle|^2$