Uncoupled DEs for a and a^+

$$\frac{da}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\left(\dot{x} + \frac{i}{m\omega}\dot{p}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{p}{m} + \frac{i}{m\omega}(-m\omega^2 x)\right)$$

$$\frac{da}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\left(-i\omega x + \frac{p}{m}\right) = -i\omega\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$$

$$\frac{da}{dt} = -i\omega a \quad \Rightarrow \quad \frac{da^+}{dt} = i\omega a^+$$

$$\Rightarrow \quad a(t) = a(0)\, e^{-i\omega t} \; ; \quad a^+(t) = a^+(0)\, e^{+i\omega t}$$

$\Rightarrow$ N is time-independent (Naturally, since H is as well)

$$\Rightarrow \quad x(t) + \frac{ip(t)}{m} = \left(x(0) + \frac{ip(0)}{m\omega}\right)e^{-i\omega t}$$

$$x(t) - \frac{ip(t)}{m} = \left(x(0) - \frac{i\,p(0)}{m\omega}\right)e^{i\omega t}$$

$$\left.\begin{aligned} \Rightarrow \quad x(t) &= x(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t) \\ p(t) &= p(0)\cos(\omega t) - m\omega x(0)\sin(\omega t) \end{aligned}\right] \text{ Just the classical EoMs.}$$

☆ Please be sure to read the bit on the Baker-Hausdorff lemma.

<u>A few Remarks on the wavefunction:</u>

$\psi(\vec{x},t) = \langle \vec{x} | \alpha, t_0; t\rangle$ we already saw that

$i\hbar \frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\,\psi(x)$ is the time-dependent

S-equation corresponding to the Hamiltonian $H = \frac{p^2}{2m} + V(x)$

The Time - Independent wave equation :

Solutions of the partial differential equation satisfied by energy eigenstates wavefunctions.

$$\langle \vec{x} | E, t_0 ; t\rangle = \langle \vec{x} | E\rangle e^{-iEt/\hbar} \Rightarrow$$ Time - dependence of a stationary state.

$t_0 = 0$ may also be an eigenstate of some other observable A that commutes with the Hamiltonian i.e. $[A, H] = 0$.

$$\Rightarrow \quad \frac{-\hbar^2}{2m}\nabla^2 \langle \vec{x} | E\rangle + V(\vec{x})\langle \vec{x} | E\rangle = E\langle \vec{x} | E\rangle$$

Typically called $u_E(\vec{x})$ rather than $\psi_E(\vec{x})$. No good reason.

What are the boundary conditions o the above PDE?

IF $E < \lim_{|\vec{x}|\to\infty} V(\vec{x})$ we need $\lim_{|\vec{x}|\to\infty} u_E(\vec{x}) = 0$ Bound States.

Otherwise we can only make conditions on the probability current for <u>scattering</u> states.

Recall $|\psi(\vec{x},t)|^2 = \rho(\vec{x},t) \leftarrow$ probability density.

Consider: $$i\hbar\, \partial_t \psi = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi$$

$$i\hbar\, \psi^* \partial_t \psi = \frac{-\hbar^2}{2m}\psi^*\nabla^2\psi + \psi^* V\psi$$

$$-i\hbar\, \psi\, \partial_t \psi^* = \frac{-\hbar^2}{2m}\psi\nabla^2\psi^* + \psi V\psi^*$$

$$i\hbar\left(\psi^*\partial_t\psi + \psi\partial_t\psi^*\right) = \frac{-\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right)$$

so $$\int d^3x\, i\hbar\left(\psi^*\partial_t\psi + \psi\partial_t\psi^*\right) = \frac{-\hbar^2}{2m}\int d^3x\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right)$$

$$i\hbar\frac{\partial}{\partial t}\int d^3x\,|\psi|^2 = -\frac{\hbar^2}{2m}\int d^3x\ \vec{\nabla}\cdot\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

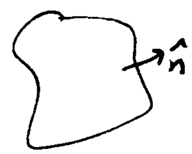
or $$\int d^3x\left\{\frac{\partial}{\partial t}\rho - \frac{i\hbar}{2m}\vec{\nabla}\cdot\vec{J}'\right\} = 0$$ ← Can do this w/o the integration.

$$\Rightarrow \quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \qquad \boxed{\vec{J} = \frac{-i\hbar}{2m}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]}$$

<u>Probability Current</u>

and $$\frac{d}{dt}\int_V d^3x\,\rho = -\int_{\partial V} d^2x\ \vec{J}\cdot\hat{n}$$

outward normal.



★ Please be sure to read the WKB stuff in Sakurai

WKB $$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - V(x)\right)\psi = 0$$

Try $\psi = \exp[iu(x)]$; let $k(x) = \sqrt{\frac{2m}{\hbar^2}(E - V(x))}$ if $E > V(x)$

$k(x) = -iK(x)$ if $E < V(x)$

Then $$i\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 + k(x)^2 = 0$$

↑ 0 for a free particle: Suggests an iteration procedure based on a small u'' assuming $V(x)$ varys slowly.

$$u_0'^2 = k^2(x) \Rightarrow \qquad 0^{\text{th}} \text{ order.}$$

$$u_0(x) = \pm \int^x k(x) + C$$

To get the next approximation:

$$u'^2 = k^2(x) + i u''$$

$(n+1)^{\text{th}}$ approximation ↑ ↖ n^{th} approximation

$$\Rightarrow \quad u_{n+1}(x) = \pm \int^x \sqrt{k^2(x) + i u_n''(x)} + C_{n+1}$$

or in the beginning we get:

$$u_1(x) = \pm \int^x \sqrt{k^2(x) \pm i k'(x)}\, dx + C_1.$$

This makes sense only if $u_1(x)$ is close to $u_0(x)$:

We need $\quad |k'(x)| << |k^2(x)|$

Then
$$u_1(x) \cong \int^x \left\{ \pm k(x) + i \frac{k'(x)}{2k(x)} \right\} dx + C_1$$

$$= \pm \int^x k(x)\,dx + i \log k(x) + C$$

$$\Rightarrow \quad \psi(x) \cong \frac{1}{\sqrt{k(x)}} e^{\pm i \int^x k(x)\,dx}$$

The condition that the WKB approx makes sense can be reformulated

$\lambda = \frac{2\pi}{k(x)}$ then $\quad |k'(x)| << |k^2(x)| \Rightarrow$

$$\cancel{\frac{2\pi}{\lambda}}\ \frac{1}{\hbar} |p'(x)| << \left| \frac{2\pi}{\lambda(x)} \right|^2 \Rightarrow$$

$$\lambda(x)\, |p'(x)| << |p(x)|$$

clearly, this breaks down at the classical turning points where $V - E \to 0$!

Propagators:

Time evolution of the wavefunction (again!)

$$|\alpha, t_0, t\rangle = e^{-iH(t-t_0)/\hbar} |\alpha, t_0\rangle$$

$$= \sum_{a'} |a'\rangle\langle a'|\alpha, t_0\rangle e^{-iE_{a'}(t-t_0)/\hbar}$$

So

$$\langle x|\alpha, t_0, t\rangle = \sum_{a'} \underbrace{\langle x|a'\rangle}_{u_{a'}(x)} \langle a'|\alpha, t_0\rangle e^{-iE_{a'}(t-t_0)/\hbar} \qquad (1)$$

$u_{a'}(x)$ ← Eigenfunctions of operator (observable) A with eigenvalue a'

Note also that $\langle a'|\alpha, t_0\rangle = \int dx \, \langle a'|x\rangle\langle x|\alpha, t_0\rangle$

$$= \int dx \, u^*_{a'}(x) \, \psi(x, t_0) = c_{a'}(t_0)$$

Expansion coefficients of the initial state in terms of the basis functions $u_{a'}(x)$.

Going back to (1) we can write this as:

Propagator

$$\psi(x, t) = \int dx' \, K(x, t; x', t_0) \, \psi(x', t_0)$$

where

$$K(x, t; x', t_0) = \sum_{a'} \langle x|a'\rangle\langle a'|x'\rangle e^{-iE_{a'}(t-t_0)/\hbar}$$

Note that: 1) $\lim_{t \to t_0} K(x, t; x', t_0) = \delta(x - x')$ ← Has to be true!