$$|a',t\rangle = |a'\rangle e^{-iE_{a'}t/\hbar}$$

picks up a phase modulation only.

in particular:

$$\langle a',t|A|a',t\rangle = a' = \langle a'|A|a'\rangle \quad \text{Time-independent}$$

$\Longrightarrow$ $[A,H]=0 \Longleftrightarrow$ A is a <u>constant of the motion</u>

In principle, you can repeat the above analysis with multiple compatible observables that commute with the Hamiltonian.

<u>The Time-Dependence of Expectation Values.</u>

Say the system starts in an eigenket of observable A. $[A,H]=0$. Now what is the time dependence of B, an observable that may not commute with either A or H?

$$|a', t_0=0; t\rangle = \mathcal{U}(t,0)|a'\rangle$$

$$\langle B\rangle(t) = \langle a', t_0=0, t|B|a', t_0=0, t\rangle$$

$$= \langle a'|\mathcal{U}^\dagger(t,0)\, B\, \mathcal{U}(t,0)|a'\rangle$$

$$= \langle a'|e^{iE_{a'}t/\hbar} B e^{-iE_{a'}t/\hbar}|a'\rangle = \langle a'|B|a'\rangle = \langle B\rangle(t=0)$$

$\langle B\rangle$ is constant as well. We call $|a'\rangle$ a <u>stationary state</u>

If you consider a superposition of stationary states:

$\langle B \rangle = ?$ when $|\alpha, t_0=0\rangle = \sum_{a'} c_{a'} |a'\rangle$

$$\Rightarrow \quad \langle B \rangle = \left\{ \sum_{a'} c_{a'}^* \langle a' | e^{iE_{a'}t/\hbar} \right\} \cdot B \cdot \left\{ \sum_{a''} c_{a''} e^{-iE_{a''}t/\hbar} |a''\rangle \right\}$$

$$\langle B \rangle = \sum_{a', a''} c_{a'}^* c_{a''} \langle a' | B | a'' \rangle \, e^{\frac{it}{\hbar}(E_{a'} - E_{a''})}$$

oscillations w/ frequency

$\omega_{a''a'} = (E_{a'} - E_{a''})/\hbar$

Correlation Amplitude and the Energy-time Uncertainty Relation

How are kets at different times related to each other.

$$|\alpha\rangle \xrightarrow[\text{Time evolution}]{} |\alpha, t\rangle$$

Correlation Amplitude: $C(t) = \langle \alpha | \alpha, t \rangle = \langle \alpha | \mathcal{U}(t,0) | \alpha \rangle$

If $|\alpha\rangle$ is an eigenket of the Hamiltonian: $|\alpha\rangle = |a'\rangle$

$$C(t) = \langle a' | \mathcal{U}(t,0) | a' \rangle = e^{-iE_{a'}t/\hbar}$$

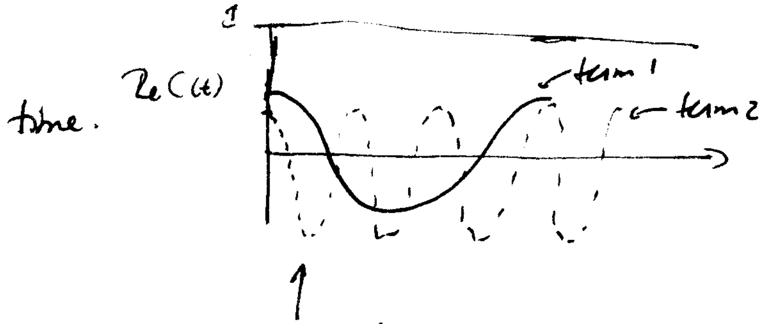
If $|\alpha\rangle$ is a superposition of such eigenkets:

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$$

$$C(t) = \sum_{a'} c_{a'}^* \langle a' | \cdot \sum_{a''} c_{a''} e^{-iE_{a''}t/\hbar} |a''\rangle$$

$$C(t) = \sum_{a'} |c_{a'}|^2 e^{-iE_{a'}t/\hbar} \; ; \text{ at } t=0 \quad C(0) = 1$$

cancellation between the terms should decrease this over



cancelation !

To understand this better, consider many eigenkets in each band of energy → quasi-continuous spectrum.

$$\sum_{a'} \rightarrow \int dE\, \rho(E)$$

states between E and E+dE = Density of States

$$c_{a'} \rightarrow g(E)\Big|_{E=E_a}$$

$\Rightarrow$

$$C(t) = \int dE\, |g(E)|^2 \rho(E)\, e^{-iEt/\hbar}$$ with the normalization condition $\int dE\, |g(E)|^2 \rho(E) = C(0) = 1$.

say $|g(E)|^2 \rho(E)$ is peaked at E_0

$$C(t) = e^{-\frac{iE_0 t}{\hbar}} \int dE\, |g(E)|^2 \rho(E)\, e^{-i(E-E_0)t/\hbar}$$

large $t \rightarrow$ oscillations kill the integral
if the interval $|E - E_0| \simeq \hbar/t$ is narrower than ΔE $\Rightarrow C(t) \rightarrow 0$

so at $t \simeq \hbar/\Delta E$ $C(t)$ starts to decay strongly.

$\Rightarrow$ $\boxed{\Delta t\, \Delta E \simeq \hbar.}$ Time Energy Uncertainty Relation

Not like x, p !

The Schrödinger vs. the Heisenberg Picture

Note we thought about time evolution as: $|\alpha\rangle \to U|\alpha\rangle$

Consider the time evolution of a matrix element:

$$\langle\beta|X|\alpha\rangle \to \langle\beta|U^{+}XU|\alpha\rangle$$

In fact, we can apply the above to every unitary operator U acting on the state kets.

There are two equivalent ways to discuss the above result.

Approach 1: $|\alpha\rangle \to U|\alpha\rangle$ operators unchanged but the state kets evolve under the unitary transformation

Approach 2:

$X \to U^{+}XU$ operators evolve under the unitary transformation but the state kets are <u>unchanged</u>.

An example from translations:

1. $|\alpha\rangle \longrightarrow \left(1 - \frac{i\vec{p}\cdot d\vec{x}'}{\hbar}\right)|\alpha\rangle$ and $\vec{x} \to \vec{x}$

2. $|\alpha\rangle \longrightarrow |\alpha\rangle$

$$\vec{x} \longrightarrow \left(1 + \frac{i\vec{p}\cdot d\vec{x}}{\hbar}\right)\vec{x}\left(1 - \frac{i\vec{p}\cdot d\vec{x}}{\hbar}\right)$$

$$\vec{x} \to \vec{x} + \frac{i}{\hbar} d\vec{x}\cdot[\vec{p},\vec{x}] = \vec{x} + d\vec{x}$$

Either way: $\langle\vec{x}\rangle \to \langle\vec{x}\rangle + \langle d\vec{x}\rangle$

Looking at the Heisenberg picture in more detail:

take $t_0 = 0$ $\quad \mathcal{U}(t) = e^{-iHt/\hbar}$

$$A^{(H)}(t) = \mathcal{U}^{+}(t)\, A^{(S)}\, \mathcal{U}(t)$$

Heisenberg operator

Schrödinger picture observable.

Recall: The state kets in the Heisenberg picture do not evolve in time: $|\alpha, t_0 = 0, t\rangle_H = |\alpha, t_0 = 0\rangle$ for all t.

Heisenberg Equation of Motion: $\quad A^{(H)} = \mathcal{U}^{+} A^{(S)} \mathcal{U}$

$$\frac{dA^{(H)}}{dt} = \frac{\partial \mathcal{U}^{+}}{\partial t} A^{(S)} \mathcal{U} + \mathcal{U}^{+} A^{(S)} \frac{\partial \mathcal{U}}{\partial t}$$

$$= -\frac{1}{i\hbar} \mathcal{U}^{+} H A^{(S)} \mathcal{U} + \mathcal{U}^{+} A^{(S)} H \mathcal{U} \frac{1}{i\hbar}$$

$$= \frac{1}{i\hbar} \left\{ -H\, \mathcal{U}^{+} A^{(S)} \mathcal{U} + \mathcal{U}^{+} A^{(S)} \mathcal{U}\, H \right\}$$

$$\boxed{\frac{dA^{(H)}}{dt} = \frac{1}{i\hbar} \left[A^{(H)}, H \right]}$$

Free Particles: Ehrenfest's Theorem

For a physical system with a classical analogue, we assume that the quantum Hamiltonian can be constructed by replacing the classical variables (x, p) by operators:

Using this principle consider a free particle

$$H = \frac{|\vec{P}|^2}{2m} \Rightarrow \quad \frac{dp_\alpha}{dt} = \frac{1}{i\hbar}[p_\alpha, H] = 0$$

The momentum is a <u>constant of the motion</u>.

In general if A is an observable such that $[A,H]=0$, $\frac{dA}{dt}=0$ — it is a constant of the motion.

What about x?

$$\frac{dx_\alpha}{dt} = \frac{1}{i\hbar}[x_\alpha, H] = \frac{1}{2mi\hbar}[x_\alpha, p^2]$$

Recalling from Chapter 1: $[x_\alpha, F(p)] = i\hbar \frac{\partial F}{\partial p_\alpha}$

$$\frac{dx_\alpha}{dt} = \frac{1}{2mi\hbar} 2p_\alpha (i\hbar) = \frac{p_\alpha}{m} \Rightarrow$$

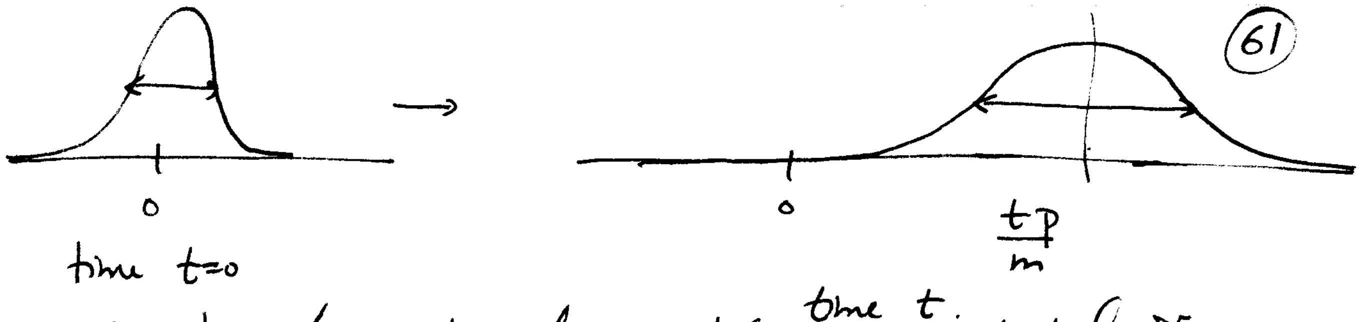
$$x_\alpha(t) = x_\alpha(0) + \frac{p_\alpha(0)}{m} t$$ — The Heisenberg operators evolve in time just like their ~~classic~~ analogous classical variables. But, this is not classical mechanics! Consider:

$$[x_\alpha(t), x_\alpha(0)] = \left[\frac{p_\alpha(0)t}{m}, x_\alpha(0)\right] = -\frac{i\hbar t}{m}$$

The x-operators at different times do not commute.

$\Rightarrow$ (Uncertainty Relation) $\quad \langle \Delta x_\alpha^2(t)\rangle \langle \Delta x_\alpha^2(0)\rangle \geq \frac{\hbar^2 t^2}{4m^2}$

The uncertainty in the particle's location grows in time



Consider the motion of a particle in a potential $V(x)$:

$$H = \frac{|\vec{p}|^2}{2m} + V(\vec{x})$$

$$\frac{dp_\alpha}{dt} = \frac{1}{i\hbar}\left[p_\alpha, V(\vec{x})\right] = -\frac{\partial V}{\partial x_\alpha}; \quad \frac{dx_\alpha}{dt} = \frac{1}{i\hbar}\left[x_\alpha, \frac{p^2}{2m}\right] = \frac{p_\alpha}{m}$$

$$\left\{\text{Using} \quad [p_\alpha, G(\vec{x})] = -i\hbar\frac{\partial G}{\partial x_\alpha}\right\}$$

And it is easy to check:

$$\frac{d^2 x_\alpha}{dt^2} = \frac{1}{i\hbar}\left[\frac{dx_\alpha}{dt}, H\right] = \frac{1}{i\hbar}\left[\frac{p_\alpha}{m}, H\right] = \frac{1}{m}\frac{dp_\alpha}{dt}$$

so $\frac{d^2 x_\alpha}{dt^2} = -\frac{1}{m}\frac{\partial V}{\partial x_\alpha} \Rightarrow m\frac{d^2 x_\alpha}{dt^2} = -\frac{\partial V}{\partial x_\alpha}$: Newton's 2nd Law

Thus:

$$\boxed{m\frac{d^2}{dt^2}\langle\vec{x}\rangle = -\langle\vec{\nabla}V(x)\rangle}$$

Ehrenfest's thm (1927)

The expectation value evolves in time just like a classical particle!

Comparing the S- and H-picture: Base kets and Transition amplitudes

S-picture: $A|a'\rangle = a'|a'\rangle$

Base ket and eigenket of observable A.

Since A does not change. The eigenvalue condition requires $|a'\rangle$ to be stationary in time as well.

H-picture: $A^{(H)}(t) \quad \mathcal{U}^{+} A(0) \mathcal{U}$

$$\mathcal{U}^{+} A(0)\, \mathcal{U}\, \mathcal{U}^{+}|a'\rangle = a' \,\mathcal{U}^{+}|a'\rangle \quad \text{or}$$

$$A^{(H)}(t)\left(\mathcal{U}^{+}|a'\rangle\right) = a'\left(\mathcal{U}^{+}|a'\rangle\right)$$

If we continue to claim that the eigenkets of A form our basis set: $\{\mathcal{U}^{+}(t)|a'\rangle\}$ must be our basis in the H-picture.

$$|a',t\rangle_H = \mathcal{U}^{+}(t)|a'\rangle \quad \text{or}$$

$$i\hbar \frac{\partial}{\partial t}|a',t\rangle_H = - H|a',t\rangle_H$$ Base kets evolve in time according to a backwards Schrödinger Eq.

Note: The eigenvalues a' do not evolve in time!

$$A^{(H)}(t) = \sum_{a'} |a',t\rangle\, a'_H \,\langle a',t| = \sum_{a'} \mathcal{U}^{+}|a'\rangle a' \langle a'|\mathcal{U}$$

$$A^{(H)}(t) = \mathcal{U}^{+}\left(\sum_{a'} |a'\rangle\langle a'| a'\right)\mathcal{U} = \mathcal{U}^{+}(t)\, A^{(s)}\, \mathcal{U}(t)$$

Of course, the expansion coefficients of a state ket in terms of base kets are the same:

$$C_{a'}(t) = \underset{\substack{\uparrow \\ \text{base ket}}}{\langle a'|} \underbrace{\left(\mathcal{U}|\alpha, t_0=0\rangle\right)}_{\text{Time-evolving state ket.}} \qquad \text{S-picture}$$

$$C_{a'}(t) = \underbrace{\left(\langle a'|\mathcal{U}(t)\right)}_{\text{base bra}} \underbrace{|\alpha, t_0=0\rangle}_{\text{Stationary state ket.}} \qquad \text{H-picture.}$$

Consider <u>Transition Amplitudes</u>:

A state is prepared at $t=0$ to be in $|a'\rangle$, $A|a'\rangle = a'|a'\rangle$
observable

At time t what is the probability amplitude for the system to be found in $|b'\rangle$ ← an eigenket of obs. B?

S-picture of Transition Amplitude:

$$\underbrace{\langle b'|}_{\text{base bra}} \underbrace{\mathcal{U}(t)|a'\rangle}_{\text{state ket}}$$

H-picture of Transition Amplitude:

$$\underbrace{\left(\langle b'|\mathcal{U}(t)\right)}_{\text{base bra}} \underbrace{|a'\rangle}_{\text{state ket}}$$

Either way

$\Longrightarrow \langle b'|\mathcal{U}(t)|a'\rangle$

Summary of S- and H-Picture.

	S	H
State ket	Moving $\lvert\alpha,t\rangle = \mathcal{U}_t\lvert\alpha\rangle$; $i\hbar\partial_t\lvert\alpha,t\rangle = H\lvert\alpha,t\rangle$	$\lvert\alpha\rangle$ Stationary
Observable	Stationary.	$A^H = \mathcal{U}^\dagger A\,\mathcal{U}$, $\frac{dA}{dt} = \frac{1}{i\hbar}[A,H]$
Base ket	stationary	Moving backwards. $\lvert a',t\rangle = \mathcal{U}^\dagger\lvert a'\rangle$ $i\hbar\,\partial_t\lvert a',t\rangle = -H\lvert a',t\rangle$

The Simple Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \qquad \omega = \sqrt{k/m}$$ natural frequency of the oscillator.

Define two non Hermitian operators:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

annihilation operator   creation operator.

$$[a,a^\dagger] = \frac{m\omega}{2\hbar}\left(-i\,[x,p]\frac{1}{m\omega} + \frac{i}{m\omega}[p,x]\right)$$

$$= \frac{-i}{2\hbar}\Big([x,p] - [p,x]\Big) = \frac{-i}{2\hbar}2(i\hbar) = 1.$$

Define the number operator: $\boxed{N = a^\dagger a}$ ← Clearly Hermitian.

Doing the multiplication:

$$a^{+}a = \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2} + \frac{i}{m\omega}[x,p] \right)$$

$$N = \frac{m\omega}{2\hbar}x^2 + \frac{1}{2\hbar}\frac{p^2}{m\omega} + \frac{1}{2\hbar}(-1)\hbar \Rightarrow$$

$$N = \frac{1}{\omega\hbar}\left(\frac{m\omega^2 x^2}{2} + \frac{p^2}{2m} \right) - \frac{1}{2} = \frac{H}{\omega\hbar} - 1/2$$

or $\boxed{H = \omega\hbar(N + 1/2)}$ ← Thus, $[N, H] = 0$

Consider the ~~eigenvalues~~ eigenkets of N: $N|n\rangle = n|n\rangle$

These are also energy eigenkets: $H|n\rangle = E_n|n\rangle$;
$E_n = \hbar\omega(n + 1/2)$

The relation between a, a^{+}, and N.

$$[N,a] = [a^{+}a, a] = a^{+}[a,a] + [a^{+},a]a$$
$$= 0 + (-1)a = -a$$
$$[N,a^{+}] = [a^{+}a, a^{+}] = a^{+}[a,a^{+}] + [a^{+},a^{+}]a$$
$$= a^{+}$$

So, what does that mean?

$$N a^{+}|n\rangle = \left([N,a^{+}] + a^{+}N \right)|n\rangle = (1 + n)a^{+}|n\rangle$$

$$\Rightarrow \quad a^{+}|n\rangle = c|n+1\rangle$$

c-number

Similarly,

$$N a|n\rangle = ([N,a] + aN)|n\rangle = (-1+n)\, a|n\rangle$$

$$\Rightarrow \quad a|n\rangle = \overbrace{c}^{\text{c-number.}}|n-1\rangle$$

How do we get these c-numbers?

$$\langle n|a^\dagger \circ a|n\rangle = \langle n|N|n\rangle = |c|^2 = n$$

so $\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$ ← Picking the phase to be 0.

Similarly $\quad a^\dagger|n\rangle = |\alpha\rangle \,; \quad \langle\alpha|\alpha\rangle = |c|^2$

$$|c|^2 = \langle n|a a^\dagger|n\rangle = \langle n|\, a^\dagger a + [a,a^\dagger]\,|n\rangle$$

$$= n+1$$

$$\Rightarrow \quad |c| = \sqrt{n+1}\, e^{i\varphi} \quad \text{again, pick } \varphi = 0.$$

$$\boxed{a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

What values of n can we take?

Suppose we keep applying a to a ket $|n\rangle$:

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^2|n\rangle = \sqrt{n(n-1)}\,|n-2\rangle$$

$$a^k|n\rangle = \sqrt{\prod_{g=0}^{k-1}(n-g)}\;|n-k\rangle$$

if $n > 0$ and $n \in \mathbb{Z}$, eventually $n - g = 0$ and we stop!
if $n \notin \mathbb{Z}$ or negative, the sequence can go on forever

but this leads to new problems!

$\langle n|N|n\rangle = \langle n|a^\dagger a|n\rangle = \langle \alpha|\alpha\rangle \geq 0$ so we can never have n negative!

$|\alpha\rangle = a|n\rangle$

The sequence of kets must stop with $n=0$. and $n \in \mathbb{Z}$.

Thus, we can identify the ground state $|0\rangle$

$$E_0 = \frac{1}{2}\hbar\omega.$$

To get the excited states, we apply the creation operator $a^\dagger$

$$|1\rangle = a^\dagger|0\rangle$$

$$|2\rangle = \frac{a^\dagger}{\sqrt{2}}|1\rangle = \frac{a^{\dagger 2}}{\sqrt{2}}|0\rangle$$

$$|3\rangle = \frac{a^\dagger}{\sqrt{3}}|2\rangle = \frac{a^{\dagger 3}}{\sqrt{2\cdot 3}}|0\rangle$$ so in general

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle \qquad E_n = \hbar\omega(n+1/2)$$

Eigenkets Energy eigenvalues.

$n = 0, 1, 2, 3, \ldots$

Note that $a^\dagger, a$ connect consecutive states $n, n+1$ or $n, n-1$.

$$\langle n'|a|n\rangle = \sqrt{n}\langle n'|n-1\rangle = \sqrt{n}\,\delta_{n',n-1}$$

$$\langle n'|a^\dagger|n\rangle = \sqrt{n+1}\langle n'|n+1\rangle = \sqrt{n+1}\,\delta_{n',n+1}$$

Considering
$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$
$$p = \frac{i\sqrt{m\hbar\omega}}{\sqrt{2}}\,(a^\dagger - a)$$
we can easily work out the matrix elements of x, p in $|n\rangle$ basis:
$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right]$$
$$\langle n'|p|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left[\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right]$$
Not diagonal. Of course x, p do not commute with N.

What do the energy eigenkets look like in position space? i.e. what are the stationary state wavefunctions?

$$a|0\rangle = 0 \implies$$
$$\langle x'|a|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x'|\left(x + \frac{ip}{m\omega}\right)|0\rangle = 0$$
This is a differential equation for $\langle x'|0\rangle$:
$$0 = \sqrt{\frac{m\omega}{2\hbar}}\left\{x'\langle x'|0\rangle + \frac{i}{m\omega}(-i\hbar)\frac{\partial}{\partial x'}\langle x'|0\rangle\right\} \quad \text{or}$$
$$0 = \left(x' + x_0^2\frac{d}{dx'}\right)\langle x'|0\rangle \quad \text{with} \quad x_0' = \sqrt{\frac{\hbar}{m\omega}}$$
$$x_0^2\frac{d}{dx'}\langle x'|0\rangle = -x'\langle x'|0\rangle \quad \text{or}$$
$$\frac{d}{dx'}\log\left(\langle x'|0\rangle\right) = -\frac{x'}{x_0^2} \implies \langle x'|0\rangle = A\,e^{-\frac{1}{2}\frac{x'^2}{x_0^2}}$$

Choose A by normalization $\Rightarrow$

$$\langle x'|0\rangle = \frac{1}{\pi^{1/4} x_0^{1/2}} e^{-\frac{1}{2}\frac{x'^2}{x_0^2}}$$

We can get all the excited state wavefunctions by applying the creation operator.

$$\langle x'|1\rangle = \langle x'|a^\dagger|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x'|\left(x - \frac{ip}{m\omega}\right)|0\rangle$$

$$\langle x'|1\rangle = \frac{1}{\sqrt{2}\,x_0}\left(x' - \frac{\hbar}{m\omega}\frac{d}{dx'}\right)\langle x'|0\rangle = \frac{1}{\sqrt{2}\,x_0}\left(x' - x_0^2\frac{d}{dx'}\right)\langle x'|0\rangle$$

$$\langle x'|2\rangle = \frac{1}{\sqrt{2!}}\langle x'|a^{\dagger 2}|0\rangle$$

$$= \frac{1}{\sqrt{2!}}\left(\frac{1}{\sqrt{2}\,x_0}\right)^2\left(x' - x_0^2\frac{d}{dx'}\right)^2\langle x'|0\rangle \quad \text{etc.}$$

In general

$$\langle x'|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}}\frac{1}{x_0^{n+1/2}}\left(x' - x_0^2\frac{d}{dx'}\right)^n e^{-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2}$$

Looking at the expectation values of x^2 and p^2 in the ground state $\Rightarrow$ Implications for the uncertainty relation

$$x^2 = \frac{\hbar}{2m\omega}\left(a^2 + a^{\dagger 2} + a^\dagger a + a a^\dagger\right)$$

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}\left(\langle 0|a^2|0\rangle + \langle 0|a^{\dagger 2}|0\rangle + \langle 0|a^\dagger a|0\rangle + \langle 0|a a^\dagger|0\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\left(0 + 0 + 0 + 1\right) = \frac{x_0^2}{2}$$

Do the same for p^2:

$$p^2 = -\frac{m\hbar\omega}{2}\left(a^{+2} + a^2 - a^+a - aa^+\right)$$

$$\langle 0|p^2|0\rangle = -\frac{m\hbar\omega}{2}(-1) = \frac{m\hbar\omega}{2}$$

Thus
$$\left.\begin{aligned} \langle 0|\frac{p^2}{2m}|0\rangle &= \frac{\hbar\omega}{4} = \frac{\langle H\rangle}{2} \\ \langle 0|\frac{m\omega^2 x^2}{2}|0\rangle &= \frac{\hbar\omega}{4} = \frac{\langle H\rangle}{2} \end{aligned}\right]$$
Agrees with the Virial theorem.

$\langle 0|x|0\rangle = \langle 0|p|0\rangle = 0$ so

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle = \frac{\hbar}{2m\omega}\ ;\ \langle(\Delta p)^2\rangle = \langle p^2\rangle = \frac{\hbar\omega m}{2}$$

The product

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4} \leftarrow$$ minimum possible uncertainty product!

For the excited states we get:

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = (n+1/2)^2\hbar^2$$ Please check this yourself.

Time-Development of an Oscillator.