$$A|a^{(l)}\rangle = a^{l}|a^{(l)}\rangle$$

so

$$UAU^{-1}U|a^{(l)}\rangle = a^{(l)}\,U|a^{(l)}\rangle$$

but $U|a^{(l)}\rangle = |b^{(l)}\rangle$ so

$$UAU^{-1}|b^{(l)}\rangle = a^{(l)}|b^{(l)}\rangle$$

same eigenvalues.

In some sense physically UAU^{-1} and A are really the same operator (observable) We will be more clear about this as we go on.

Continuous Spectra: Position Momentum, and Translation –

Generally there are subtle issues here. We will avoid most of them.

$$\xi|\xi'\rangle = \xi'|\xi'\rangle$$

eigenket

continuous spectrum of ξ'

eigenvalue

operator w/ continuous spectrum

$$\langle a'|a''\rangle = \delta_{a',a''} \longrightarrow \langle\xi'|\xi''\rangle = \delta(\xi'-\xi'')$$

Dirac δ-function

$$\sum_{a'}|a'\rangle\langle a'| = \mathbb{1} \longrightarrow \int d\xi'\,|\xi'\rangle\langle\xi'| = \mathbb{1}$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \longrightarrow |\alpha\rangle = \int d\xi' \, |\xi'\rangle\langle \xi'|\alpha\rangle$$

$$\sum_{a'} |\langle a'|\alpha\rangle|^2 = 1 \longrightarrow \int d\xi' \, |\langle \xi'|\alpha\rangle|^2 = 1$$

$$\langle \beta|\alpha\rangle = \sum_{a'} \langle \beta|a'\rangle\langle a'|\alpha\rangle \longrightarrow \langle \beta|\alpha\rangle = \int d\xi' \, \langle \beta|\xi'\rangle\langle \xi'|\alpha\rangle$$

$$\langle a''|A|a'\rangle = a' \delta_{a',a''} \longrightarrow \langle \xi''|\xi|\xi'\rangle = \xi' \delta(\xi'' - \xi')$$

Position Eigenkets and Position Measurements

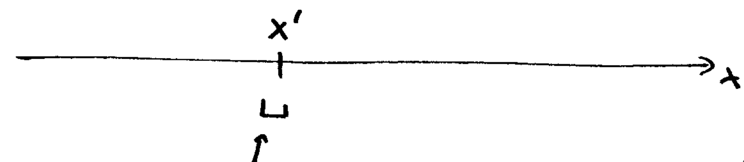
Position eigenket: $|x'\rangle$: $x|x'\rangle = x'|x'\rangle$

operator c-number

Consider an arbitrary ket:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x'|\alpha\rangle$$

Position measurement



detector ← can tell when the particle is in the range: $(x' - \Delta/2 \, , \, x' + \Delta/2)$

When the detector goes off:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'' \, |x''\rangle\langle x''|\alpha\rangle \xrightarrow[\text{measurement}]{} \int_{x'-\Delta/2}^{x'+\Delta/2} dx'' \, |x''\rangle\langle x''|\alpha\rangle$$

In the limit that $\Delta \to 0$ probability of a detection event is:

$$|\langle x'|\alpha\rangle|^2 \, dx'$$

$$\Delta = dx' \to 0$$

Probability of the particle being somewhere is unity:

$$\int_{-\infty}^{\infty} dx' \, |\langle x'|\alpha\rangle|^2 = \int_{-\infty}^{\infty} \langle\alpha|x'\rangle\langle x'|\alpha\rangle dx' = \langle\alpha|\alpha\rangle = 1$$

$\langle x'|\alpha\rangle$ = wavefunction

Extension to 3D $\quad \langle\vec{x}'|\alpha\rangle \, ; \quad |\alpha\rangle = \int d^3\vec{x}' \, |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$

Note: $|\vec{x}'\rangle$ = simultaneous eigenket of the operators x, y, z. We have thus assumed that $[x_i, x_j] = 0 \quad i,j = x,y,z$.

Translation:

$$\mathcal{J}(d\vec{x}')\,|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle$$

$\leftarrow$ moves the localized state at $|\vec{x}'\rangle$ to one localized at $\vec{x}' + d\vec{x}'$

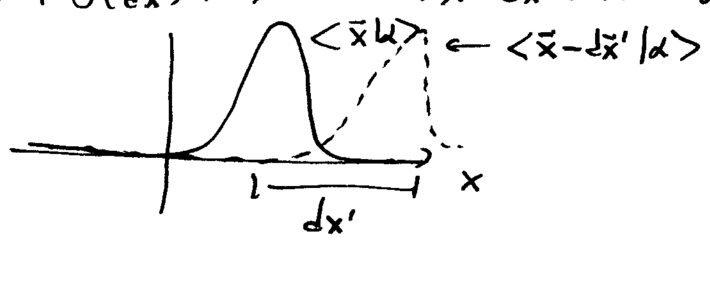
$$|\alpha\rangle \to \mathcal{J}(d\vec{x}')|\alpha\rangle = \mathcal{J}(d\vec{x}')\int d^3\vec{x}' \, |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$$

$$= \int d^3\vec{x}' \, |\vec{x}' + d\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle = \int d^3\vec{x}' \, |\vec{x}'\rangle\langle\vec{x}' - d\vec{x}'|\alpha\rangle$$

so if we call

$\langle\vec{x}|\alpha\rangle$ the position-space wavefunction

$\langle\vec{x}\,|\,\mathcal{J}(d\vec{x}')\,|\alpha\rangle = \langle\vec{x} - d\vec{x}'|\alpha\rangle$ is the translated wavefunction



Properties of the infinitesimal translation operator.

(i) Conservation of probability

$$\langle\alpha|\alpha\rangle = 1 = \langle\alpha|\mathcal{T}^{\dagger}(d\vec{x}')\,\mathcal{T}(d\vec{x}')|\alpha\rangle .$$

$$\Rightarrow \quad \mathcal{T}^{\dagger}(d\vec{x}')\,\mathcal{T}(d\vec{x}') = \mathbb{1} \quad \leftarrow \text{unitary operator.}$$

(ii) Two successive translations is equivalent to a single translation by the sum of the displacements

$$\mathcal{T}(d\vec{x}')\,\mathcal{T}(d\vec{x}'') = \mathcal{T}(d\vec{x}' + d\vec{x}'')$$

(iii) $$\mathcal{T}^{-1}(d\vec{x}') = \mathcal{T}(-d\vec{x}')$$

(iv) $$\lim_{d\vec{x}' \to 0} \mathcal{T}(d\vec{x}') = \mathbb{1} . \iff \mathbb{1} - \mathcal{T}(d\vec{x}') \propto \mathcal{O}(d\vec{x})$$

If we take $$\boxed{\mathcal{T}(d\vec{x}') = \mathbb{1} - i\,\vec{K}\cdot d\vec{x}'}$$

Hermitian operators

we satisfy all four required properties.

$$\mathcal{T}^{\dagger}(d\vec{x}')\,\mathcal{T}(d\vec{x}') = \left(1 + i\,\vec{K}^{\dagger}\cdot d\vec{x}'\right)\left(1 - i\,\vec{K}\cdot d\vec{x}'\right)$$

$$= 1 + i(\vec{K}^{\dagger} - \vec{K})\cdot d\vec{x}' + \mathcal{O}(dx^2)$$

$$= 1 + \mathcal{O}(dx^2)$$ unitary for infinitesimal translations.

Similarly: $$\mathcal{T}(d\vec{x}')\,\mathcal{T}(d\vec{x}'') = 1 - i\,\vec{K}\cdot(d\vec{x}' + d\vec{x}'') + \mathcal{O}(dx^2)$$

$$= \mathcal{T}(d\vec{x}' + d\vec{x}'')$$

What is the relation between $\vec{K}$ and $\vec{x}$?

$$\vec{x}\,\mathcal{J}(d\vec{x}')|\vec{x}'\rangle = \vec{x}\,|\vec{x}'+d\vec{x}'\rangle = (\vec{x}'+d\vec{x}')\,|\vec{x}'+d\vec{x}'\rangle$$

$$\mathcal{J}(d\vec{x}')\,\vec{x}\,|\vec{x}'\rangle = \vec{x}'\,|\vec{x}'+d\vec{x}'\rangle$$

so $\quad [\vec{x}, \mathcal{J}(d\vec{x}')]\,|\vec{x}'\rangle = d\vec{x}'\,|\vec{x}+d\vec{x}'\rangle = d\vec{x}'\,|\vec{x}\rangle + \mathcal{O}(dx^2)$

$\Rightarrow$ The operator equality

c-number $d\vec{x}' \times \mathbb{1}$ ← identity operator

$$[\vec{x}, \mathcal{J}(d\vec{x}')] = d\vec{x}' \Rightarrow$$

$$-i\,\vec{x}\,\vec{K}\cdot d\vec{x}' + i\,\vec{K}\cdot d\vec{x}'\,\vec{x} = d\vec{x}'$$

put $d\vec{x}'$ in the j^{th} direction and dot the above into the i^{th} direction

$$-i\,x_i K_j + i\,K_j x_i = \delta_{ij} \quad \text{or}$$

$$\boxed{[x_i, K_j] = i\,\delta_{ij}}$$

c-number (0 or 1) × $\mathbb{1}$. ← identity operator.

Momentum as a Generator of Translation

We can identify $\vec{K}$ as $\dfrac{\vec{P}}{\hbar}$ ← momentum. ($\hbar$ ← action $[\mathcal{L}dt]$)

$$\Rightarrow \quad \mathcal{J}(d\vec{x}') = \mathbb{1} - \frac{i\,\vec{p}\cdot d\vec{x}}{\hbar}$$

so

$$[x_i, p_j] = i\hbar\,\delta_{ij}$$

Note: our uncertainty relation implies $\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \frac{1}{4}\hbar^2$

W. Heisenberg.

How to go from infinitesimal translations to finite ones

Successive infinitesimal translations create a finite one.

$$\mathcal{T}(\overbrace{\Delta x'\,\hat{x}}^{\Delta\vec{x}'})\,|\vec{x}'\rangle = |\Delta\vec{x}' + \vec{x}'\rangle$$

$\uparrow$

Finite Translation

N steps of length $|\Delta\vec{x}|/N$. and let $N \to \infty$.

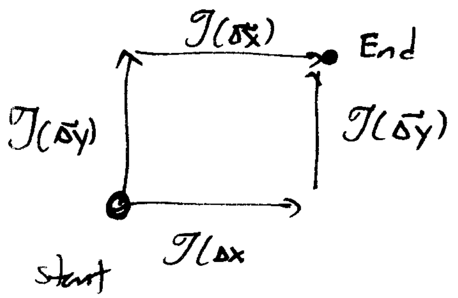
move in $\hat{x}$ direction

$$\Rightarrow \quad \mathcal{T}(\Delta\vec{x}') = \lim_{N\to\infty}\left(1 - \frac{i p_x \Delta x'}{\hbar N}\right)^N = e^{-i p_x \Delta x'/\hbar}$$

We understand an exponentiated operator in terms of its Taylor series - just like for any analytic function of an operator.

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The Abelian nature of the group of translations and the commutation relations of momentum.



$$\Rightarrow \quad \mathcal{T}(\Delta\vec{x})\,\mathcal{T}(\Delta\vec{y}) = \mathcal{T}(\Delta\vec{y})\,\mathcal{T}(\Delta\vec{x})$$

$\Rightarrow$ Translation group is Abelian.

Working out the implications for the generators:

$$[\mathcal{J}(\Delta \vec{y}'), \mathcal{J}(\Delta \vec{x}')] = \left[1 - \frac{i p_y \Delta y'}{\hbar} - \frac{p_y^2 (\Delta y')^2}{\hbar^2} \cdots, 1 - i\frac{p_x \Delta x'}{\hbar} - \frac{p_x^2 (\Delta x')^2}{\hbar^2} + \cdots\right]$$

$$\simeq \frac{(\Delta x')(\Delta y')}{\hbar^2}[p_y, p_x] + \mathcal{O}((\Delta x)^3)$$

$\Rightarrow$ $[p_x, p_y] = 0$ or in general $\boxed{[p_i, p_j] = 0}$

$\Rightarrow$ p_x, p_y, p_z are mutually compatible observables.

What is the effect of translation on a momentum eigenket?

$$\mathcal{J}(d\vec{x}')|\vec{p}'\rangle = \left(1 - \frac{i\vec{p}\cdot d\vec{x}'}{\hbar}\right)|\vec{p}'\rangle = \left(\frac{\hbar - i\vec{p}'\cdot d\vec{x}'}{\hbar}\right)|\vec{p}'\rangle$$

changes the phase of $|\vec{p}'\rangle$ but that is all.

In fact $|\vec{p}'\rangle$ is an eigenket of $\mathcal{J}(d\vec{x}')$ w/ eigenvalue

$\left(1 - \frac{i\vec{p}'\cdot d\vec{x}}{\hbar}\right)$. ← Why is this complex? Ans $\mathcal{J}(d\vec{x}')$ is unitary, not Hermitian.

This shouldn't be surprising since $[\vec{p}, \mathcal{J}(d\vec{x})] = 0$.

Summary of Canonical Commutation Rules:

$[x_i, x_j] = 0 \quad [p_i, p_j] = 0 \quad [x_i, p_j] = i\hbar\delta_{ij}$

P. A. M. Dirac "fundamental quantum conditions"

Algebraic Properties of commutators.

$[A, A] = 0, \quad [A, B] = -[B, A]$

$[A, c] = 0 \quad c = c\text{-number}$

$[A + B, C] = [A, C] + [B, C]$