Unitary $\iff$ $U^+U = \mathbb{1}$. i.e. $U^+ = U^{-1}$

Pf: By construction $U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|$

clearly $U|a^{(l)}\rangle = |b^{(l)}\rangle$ and

$$U^+U = \sum_{ij} |a^{(i)}\rangle\langle b^{(i)}|\,|b^{(j)}\rangle\langle a^{(j)}|$$

$$= \sum_i |a^{(i)}\rangle\langle a^{(i)}| = \mathbb{1}. \qquad \blacksquare$$

The transformation matrix: In other words the matrix representation of U.

In the (old) $|a^{(i)}\rangle$ basis

$$U \doteq U_{kl} = \langle a^{(k)}|U|a^{(l)}\rangle = \langle a^{(k)}|b^{(l)}\rangle$$

Given a ket $|\alpha\rangle$ in the old basis, it has the expansion

$$|\alpha\rangle = \sum_k |a^{(k)}\rangle\langle a^{(k)}|\alpha\rangle .$$

In the new basis $|b^{(l)}\rangle$ it is:

$$\cancel{U|\alpha\rangle = \sum_{k,l} |b^{(l)}\rangle\langle a^{(l)}|a^{(k)}\rangle\langle a^{(k)}|\alpha\rangle}$$

$$\cancel{= \sum_l |b^{(l)}\rangle\langle a^{(l)}|\alpha\rangle}$$

What are the coeffecients of $|\alpha\rangle$ in a $|b^{(\ell)}\rangle$ expansion?

$$\langle b^{(\ell)}|\alpha\rangle = \sum_k \langle b^{(\ell)}|a^{(k)}\rangle\langle a^{(k)}|\alpha\rangle$$

$$= \sum_k \langle a^{(k)}|b^{(\ell)}\rangle^* \langle a^{(k)}|\alpha\rangle$$

$$= \sum_k U^+_{\ell k}\langle a^{(k)}|\alpha\rangle = \sum_k \langle a^{(\ell)}|U^+|a^{(k)}\rangle\langle a^{(k)}|\alpha\rangle$$

so

$$\underbrace{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}_{\doteq\, |\alpha\rangle} = \underbrace{\begin{pmatrix} & U^+ & \end{pmatrix}}_{\doteq\, U^+}\underbrace{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}_{\doteq\, |\alpha\rangle}$$

$|\alpha\rangle$ new coordinates i.e. in B-ket basis

U^+ in A-ket basis

$|\alpha\rangle$ old coordinates i.e. in A-ket basis

How does the matrix representation of an operator change going from the A-basis to the B-basis?

$$\underbrace{\langle b^{(k)}|X|b^{(\ell)}\rangle}_{\doteq\, \tilde{X}_{k\ell}} = \sum_{ij} \underbrace{\langle b^{(k)}|a^{(j)}\rangle}_{\doteq\, U^+_{kj}}\underbrace{\langle a^{(j)}|X|a^{(i)}\rangle}_{\doteq\, X_{ji}}\underbrace{\langle a^{(i)}|b^{(\ell)}\rangle}_{\doteq\, U_{i\ell}}$$

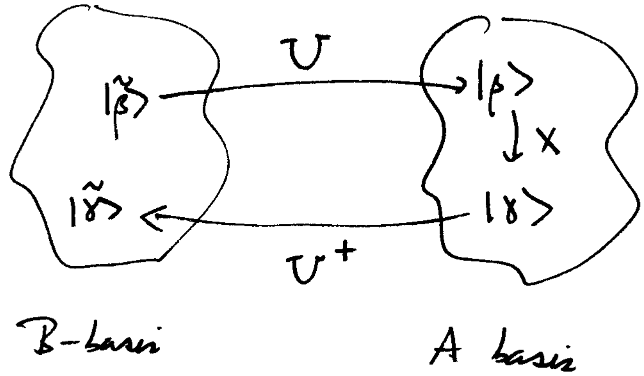
$\tilde{X}_{k\ell}$ in B-basis

X_{ji} in A-basis

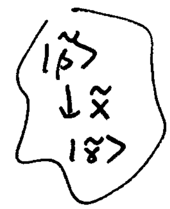
so

$$\Longrightarrow \boxed{\tilde{X}_{k\ell} = U^+_{kj}\, X_{ji}\, U_{i\ell}}$$

This makes sense



define the action of $\tilde{X}$



$\tilde{X}|\tilde{\beta}\rangle = |\tilde{\gamma}\rangle$ and $X|\beta\rangle = |\gamma\rangle$ Similarity Transform.

Note that the trace of an operator X

$$Tr(X) = \sum_{a'} \langle a'|X|a'\rangle$$ <u>is independent of representation</u>. Prove this to yourself.

Unitary Equivalent Operators

<u>Thm</u>: Consider two orthonormal bases $\{|a'\rangle\}$ and $\{|b'\rangle\}$ connected by the unitary operator U

$$U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|$$

We may construct a unitary transform of A, UAU^{-1}

$$A \longrightarrow UAU^{-1}$$

are called <u>unitary equivalent observables</u>.

They have the same eigenvalues i.e. <u>spectra</u>.