e.

We can define raising and lowering operators:

$$S_\pm = S_x \pm i S_y$$

$$S_+ = \frac{\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +| + |+\rangle\langle -| - |-\rangle\langle +| \right]$$

$$S_+ = \hbar\, |+\rangle\langle -| \quad \leftarrow \text{Non Hermitian}$$

$$S_+|+\rangle = 0 \qquad S_+|-\rangle = \hbar\,|+\rangle$$

Similarly $\quad S_- = \hbar\, |-\rangle\langle +|$

Compatible Observables: -

A, B compatible $\Leftrightarrow [A,B] = 0$

Consider two compatible observables A and B

ket space spanned by A. But it is also spanned by B.

We will be able to label all the kets by a set of basis kets

$|a, b\rangle$

$\uparrow \uparrow$

eigenvalues of A and B respectively.

* Note we now can admit the idea of degeneracy -

more than one state can have the same eigenvalue b

$$B\,|a', b\rangle = b\,|a', b\rangle$$
$$B\,|a'', b\rangle = b\,|a'', b\rangle$$

Generally, we can uniquely label states by a small set of eigenvalues of compatible operators.

Thm: A, B are Hermitian operators $[A,B]=0$. The eigenvalues of A are not degenerate. Then the matrix elements $\langle a''|B|a'\rangle$ are all diagonal.

Pf:

$\langle a''|[A,B]|a'\rangle = 0$ and so we have

$$\langle a''|[A,B]|a'\rangle = \langle a''|AB-BA|a'\rangle = (a''-a')\langle a''|B|a'\rangle = 0$$

$\Rightarrow$ if $a'' \neq a'$ then $\langle a''|B|a'\rangle = 0$

Thus Both A and B can be simultaneously represented by diagonal matrices.

Note: $$B = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|$$

apply this to an eigenket of A $|a'\rangle$

$$B|a'\rangle = \sum_{a''} \delta_{a''a'} |a''\rangle\langle a''|B|a'\rangle = \langle a'|B|a'\rangle |a'\rangle$$

so $B|a'\rangle = b|a'\rangle$ where $b = \langle a'|B|a'\rangle$

Eigenvector condition Eigenvalue.

$\Rightarrow$ $|a'\rangle$ is a simultaneous eigenket of A and B.

Actually, this proof holds if the eigenkets of A have an n-fold degeneracy: $A|a'^{(k)}\rangle = a'|a'^{(k)}\rangle \quad k=1,\ldots,n$

In each degenerate subspace $\{|a'^{(i)}\rangle, i=1,\dots,n_i\}$ you need to take the appropriate linear combos to diagonalize the B operator.

Here is a nearly trivial example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \leftarrow \text{2 degeneracy eigenvalues.}$$

eigenvectors $|x\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|y\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|z\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$A|x\rangle = +1\,|x\rangle \qquad A\begin{Bmatrix} |y\rangle \\ |z\rangle \end{Bmatrix} = -1\begin{Bmatrix} |y\rangle \\ |z\rangle \end{Bmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & b \\ 0 & -b & b \end{pmatrix} ; \qquad [A,B] = 0 \quad \text{(check this!)}$$

eigenvalues of B $\{1, (1-i)b, (1+i)b\}$

But we can take the eigenvectors to be: $|x\rangle$

$$|1\rangle = \frac{1}{\sqrt{2}}\left(i|y\rangle + |z\rangle\right)$$
$$|2\rangle = \frac{1}{\sqrt{2}}\left(-i|y\rangle + |z\rangle\right)$$

↑ linear combos of $|a'\rangle$ in the degenerate subspace of $\{|y\rangle, |z\rangle\}$.

$\Longrightarrow$ The set of kets:

$$|x\rangle, \quad |1\rangle = \frac{1}{\sqrt{2}}\left(i|y\rangle + |z\rangle\right)$$
$$|2\rangle = \frac{1}{\sqrt{2}}\left(-i|y\rangle + |z\rangle\right)$$

form a basis in which A and B are diagonal.

Note $\langle 1|2\rangle = \frac{1}{2}\left[\langle z| - i\langle y|\right]\left[|z\rangle - i|y\rangle\right]$

$= \frac{1}{2}\left[1 - 1\right] = 0$ So this is an orthonormal basis as well.

degenerate A eigenvalue

We could label the states as

$|x\rangle = |1, 1\rangle$. $|1\rangle \rightarrow |-1, (1-i)b\rangle$; $|2\rangle \rightarrow |-1, (1+i)b\rangle$

A B eigenvalue

different B eigenvalues.

To follow Sakurai's notation more closely

Collective Index.

If we have a <u>maximal</u> set of commuting observables:

$$[A,B] = [B,C] = [A,C] = \cdots = 0$$

The eigenvalues of A, B, C specify uniquely each eigenket.

Then we have orthonormality

$$\langle a, b, \ldots | a', b', \ldots\rangle = \delta_{aa'}\delta_{bb'}\cdots$$

and completeness:

$$\sum_{a', b', c', \ldots} |a', b', c', \ldots\rangle\langle a', b', c', \ldots| = \mathbb{1}.$$

Now if you have a quantum state you can measure A and get a'. Then you can measure B and get b' — BUT you still know that if you measure A again you will get the <u>same</u> result.

$$|\alpha\rangle \xrightarrow[\text{A measurement}]{} |a',b'\rangle \xrightarrow{\text{B measurement}} |a',b'\rangle \xrightarrow[\text{A measurement}]{} |a',b'\rangle$$

If A has degenerate eigenvalues:

$$|\alpha\rangle \xrightarrow[\text{A measurement}]{} \sum_{i=1}^{n} c_{a'}^{(i)} |a', b^{(i)}\rangle$$

n-dimensional degenerate subspace of kets: all have the same a' eigenvalue, but different eigenvalues. $b^{(i)}$ $i=1,\dots,n$

The measurement of B will select one of these $b^{(j)}$ $1<j<n$

$$|\alpha\rangle \xrightarrow[A]{} \sum_{i=1}^{n} c_{a'}^{(i)} |a', b^{(i)}\rangle \xrightarrow[B]{} |a', b^{(j)}\rangle$$

with probability $|c_{a'}^{(j)}|^2$.

The third measurement of A will still give you a' back

$$|\alpha\rangle \xrightarrow[A]{} \sum_{i=1}^{n} c_{a'}^{(i)} |a', b^{(i)}\rangle \xrightarrow[B]{} |a', b^{(j)}\rangle \xrightarrow[A]{} |a', b^{(j)}\rangle$$

A and B measurements do not interfere with each other.

Incompatible Observables:

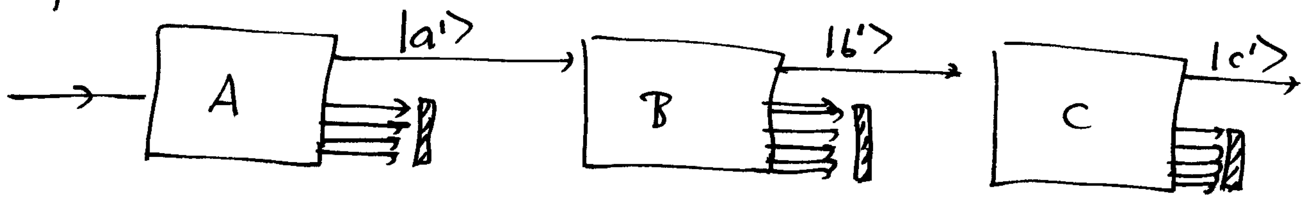
A, B are incompatible $\Rightarrow$ They do not have a complete set of simultaneous eigenkets

Pf: Assume they do $\{ |a', b'\rangle \}$ ← complete set of simultaneous eigenkets

$$AB|a',b'\rangle = a'b'|a',b'\rangle$$
$$BA|a',b'\rangle = a'b'|a',b'\rangle$$
$$\Rightarrow [A,B]|a',b'\rangle = 0$$ and so it must for any linear combination of such kets. $\Rightarrow [A,B]=0$, ✓ Contradiction.

Note: There may be a subspace for which $[A,B]|a',b'\rangle = 0$ holds.

Incompatible Observables and the Generalized Stern-Gerlach Experiment:



What is the prob of getting $|c'\rangle$ when the beam coming out of the A-filter is normalized?

Ans: $|\langle b'|a'\rangle|^2 \times |\langle c'|b'\rangle|^2$

prob of getting through the B-filter

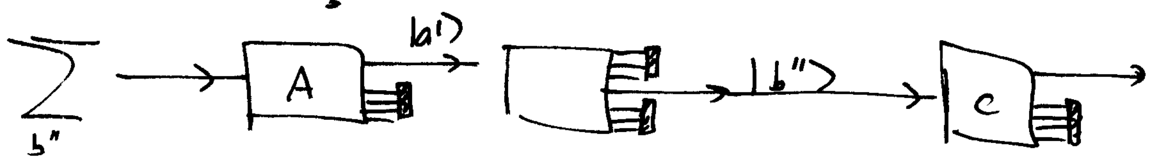
Prob of getting through the C-filter.

Now sum over all b' to get the total probability for going through all routes to the C-filter.

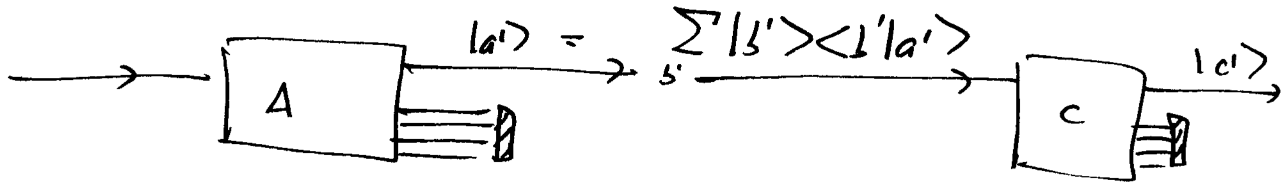
In order words, we move the "hole" in the B-filter sequentially to obtain the prob of going through each of the $|b'\rangle$ states.

$$\sum_{b'} |\langle c'|b'\rangle|^2 \; |\langle b'|a'\rangle|^2$$

$$= \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle \qquad (1)$$



Now compare this result to a simpler experiment where we just <u>remove</u> the B-filter:



$$\text{Prob} = |\langle c'|a'\rangle|^2 = \left|\sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\right|^2$$

$$= \sum_{b'b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle \quad (2)$$

Note that (1) and (2) are different!

Mathematically, this is due to where we took the absolute square. ←

Physically, the act of measuring B changes the state of the system.

The key point is that measuring B and not measuring B become equivalent only if $[A,B]=0$

or

$[B,C]=0$.

More on this later...

The Uncertainity Relation:

For an observable A, we define the operator

$$\Delta A = A - \langle A \rangle$$

taken for the physical state under consideration

The expection value of $(\Delta A)^2 \Leftarrow$ Dispersion of A

$$\begin{aligned}\langle (\Delta A)^2 \rangle &= \langle A^2 - 2A\langle A\rangle + \langle A\rangle^2 \rangle \\ &= \langle A^2\rangle - 2\langle A\rangle^2 + \langle A\rangle^2 \\ &= \langle A^2 \rangle - \langle A\rangle^2\end{aligned}$$

— also called the variance or mean square deviation

This characterizes the "fuzziness" of a given measurement.

The uncertainity Relation:

For two observables A, B

$$\boxed{\langle (\Delta A)^2\rangle\langle (\Delta B)^2\rangle \geq \frac{1}{4}|\langle [A,B]\rangle|^2}$$

To prove this we consider first three lemmas:

Lemma 1: The Schwartz Inequality

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

Pf: Consider the ket $|\psi\rangle = |\alpha\rangle + \lambda|\beta\rangle$

$\langle\psi|\psi\rangle \geq 0$ so

$[\langle\alpha| + \lambda^*\langle\beta|][|\alpha\rangle + \lambda|\beta\rangle] \geq 0$ for any c-number λ.

$$\langle\alpha|\alpha\rangle + |\lambda|^2\langle\beta|\beta\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle \geq 0$$

Now choose λ to minimize the LHS.

$\lambda = re^{i\varphi}$ and call $\langle\alpha|\beta\rangle = c \leftarrow$ some c-number

$$\langle\alpha|\alpha\rangle + r^2\langle\beta|\beta\rangle + re^{i\varphi}c + re^{-i\varphi}c^* = f(r,\varphi)$$

$$\frac{\partial f}{\partial r} = 0 \Rightarrow 2r\langle\beta|\beta\rangle + ce^{i\varphi} + c^*e^{-i\varphi} = 0$$

take $c = |c|e^{i\alpha}$ then $r = \dfrac{-|c|(e^{i(\varphi+\alpha)} + e^{-i(\varphi+\alpha)})}{2\langle\beta|\beta\rangle}$

Note $\dfrac{\partial^2 f}{\partial r^2} = \langle\beta|\beta\rangle > 0$ so this is a min.

$$\frac{\partial f}{\partial\varphi} = 0 \Rightarrow r|c|i[e^{i(\varphi+\alpha)} - e^{-i(\varphi+\alpha)}] = 0$$

$$\Rightarrow \alpha+\varphi = 0, \pi$$

Take this one.

$$\frac{\partial^2 f}{\partial\varphi^2} = -r|c|[e^{i(\varphi+\alpha)} + e^{i(\varphi+\alpha)}] \longrightarrow \begin{cases} > 0 & \text{if } \alpha+\varphi = \pi \checkmark \\ < 0 & \text{if } \alpha+\varphi = 0 \end{cases}$$

So,

$$r = \frac{-|c|\,2\cos\pi}{2\langle\beta|\beta\rangle} = \frac{|c|}{\langle\beta|\beta\rangle}$$

and $\lambda = \frac{|c|}{\langle\beta|\beta\rangle} e^{i(\pi-\alpha)} = \frac{|c| e^{-i\alpha}(-1)}{\langle\beta|\beta\rangle} = \frac{-c^*}{\langle\beta|\beta\rangle} = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$

Then we have

$$\langle\alpha|\alpha\rangle + \frac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle} + -\frac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle} - \frac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle} \geq 0$$

or

$$\langle\alpha|\alpha\rangle - \frac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle} \geq 0 \Rightarrow \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\beta|\alpha\rangle|^2$$

Lemma 2: The expectation value of a Hermitian operator is real.

Recall $\langle a''|B|a'\rangle = \langle a'|B|a''\rangle^*$ and now set $a' = a''$.

Lemma 3: The expectation value of an anti-Hermitian operator is imaginary

$C^\dagger = -C$

See the proof of lemma two!

Now, to prove the uncertainity relation:

Consider any ket $|\psi\rangle$

$|\alpha\rangle = (\Delta A)|\psi\rangle$, ΔA and ΔB are Hermitian so:

$|\beta\rangle = (\Delta B)|\psi\rangle$

From Lemma 1. $\langle\psi|(\Delta A)^2|\psi\rangle\langle\psi|(\Delta B)^2|\psi\rangle \geq |\langle\psi|\Delta A\,\Delta B|\psi\rangle|^2$

Or,

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle \Delta A\, \Delta B\rangle|^2$$

Rewrite the RHS $\quad \Delta A\, \Delta B = \underbrace{\tfrac{1}{2}[\Delta A, \Delta B]}_{\text{Anti-Hermitian}} + \underbrace{\tfrac{1}{2}\{\Delta A, \Delta B\}}_{\text{Hermitian.}}$

$\Rightarrow$

$$\langle \Delta A\, \Delta B\rangle = \underbrace{\tfrac{1}{2}\langle[\Delta A, \Delta B]\rangle}_{\text{Imaginary}} + \underbrace{\tfrac{1}{2}\langle\{\Delta A, \Delta B\}\rangle}_{\text{Real}}$$

so

$$|\langle \Delta A\, \Delta B\rangle|^2 = \frac{1}{4}|\langle[A,B]\rangle|^2 + \frac{1}{4}|\langle\{\Delta A, \Delta B\}\rangle|^2$$

this is pos. def.

so

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2 \quad \checkmark$$

Change of Basis:

If we have two incompatible observables A, B, the ket space can be spanned by eigenkets of either one $\{|a'\rangle\}$ or $\{|b'\rangle\}$.

How are the two descriptions of the space related?

<u>Thm</u>: Given two sets of base kets (both orthonormal and complete) $\exists$ a unitary operator $U \ni$

$$|b^{(i)}\rangle = U|a^{(i)}\rangle \qquad i = 1, \ldots, N$$

Unitary $\iff$ $U^{+}U = \mathbb{1}$. i.e. $U^{+} = U^{-1}$

Pf: By construction $U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|$

clearly $U|a^{(l)}\rangle = |b^{(l)}\rangle$ and

$$U^{+}U = \sum_{ij} |a^{(i)}\rangle\langle b^{(i)}|\,|b^{(j)}\rangle\langle a^{(j)}|$$

$$= \sum_{i} |a^{(i)}\rangle\langle a^{(i)}| = \mathbb{1}.$$

The transformation matrix: In other words the matrix representation of U.

In the (old) $|a^{(i)}\rangle$ basis

$$U \doteq U_{kl} = \langle a^{(k)}|U|a^{(l)}\rangle = \langle a^{(k)}|b^{(l)}\rangle$$

Given a ket $|\alpha\rangle$ in the old basis, it has the expansion

$$|\alpha\rangle = \sum_k |a^{(k)}\rangle\langle a^{(k)}|\alpha\rangle.$$

In the new basis $|b^{(l)}\rangle$ it is:

$$\cancel{U|\alpha\rangle = \sum_{k,l} |b^{(l)}\rangle\langle a^{(l)}|a^{(k)}\rangle\langle a^{(k)}|\alpha\rangle}$$

$$\cancel{= \sum_{l} |b^{(l)}\rangle\langle a^{(l)}|\alpha\rangle}$$