Since $\quad \langle\beta|X|\alpha\rangle = \langle\alpha|X^\dagger|\beta\rangle^*$

$$\langle a''|X|a'\rangle^* = \langle a'|X^\dagger|a''\rangle$$

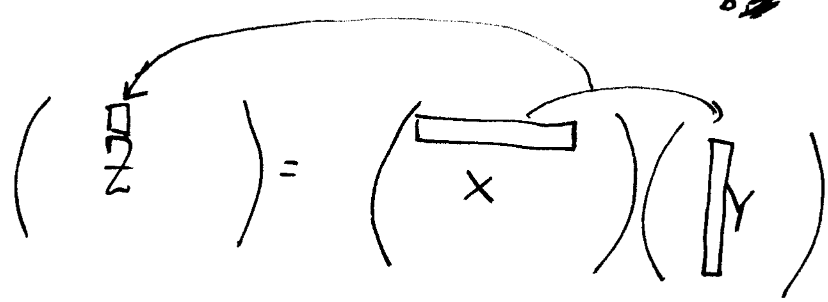
Hermitian adjoint operator is represented by the complex conjugate transpose of the matrix representing the operator.

This representation of the operator means that operator multiplication is implemented by usual matrix multiplication

$$Z = XY$$

$$\langle a''|Z|a'\rangle = \langle a''|XY|a'\rangle = \sum_{b} \langle a''|X|b\rangle\langle b|Y|a'\rangle$$

so



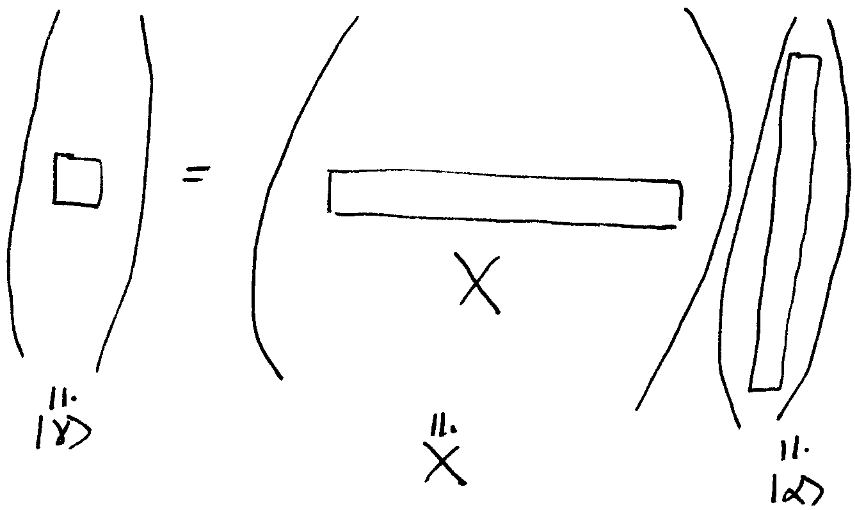
Similarly, operators act on kets by matrix multiplication

$$|\gamma\rangle = X|\alpha\rangle$$

$$\langle a'|\gamma\rangle = \langle a'|X|\alpha\rangle = \sum_b \langle a'|X|b\rangle\langle b|\alpha\rangle$$

$$|\gamma\rangle \doteq \begin{pmatrix} \langle a^1|\gamma\rangle \\ \langle a^2|\gamma\rangle \\ \vdots \\ \langle a^N|\gamma\rangle \end{pmatrix} \quad \text{then}$$

$$|\gamma\rangle = X|\alpha\rangle$$



$\doteq$ ← is represented by

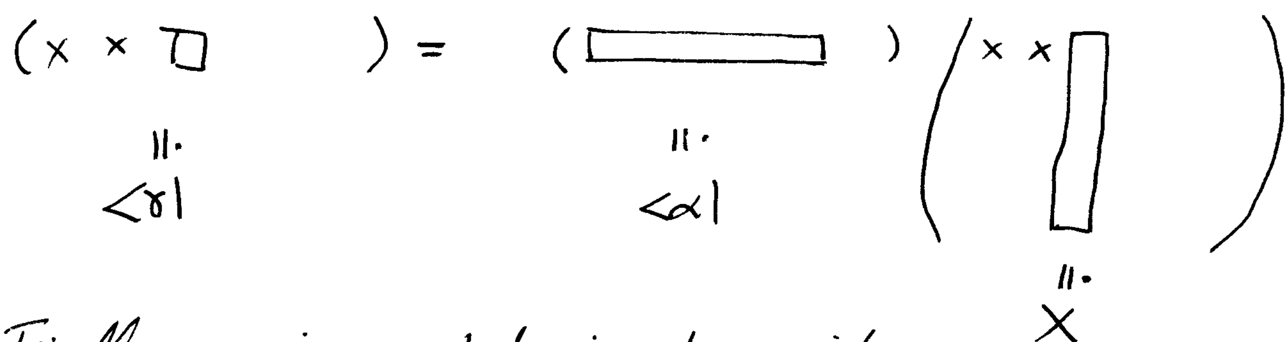
What about bras?

$$\langle\gamma| = \langle\alpha|X$$

$$\langle\gamma|a'\rangle = \langle\alpha|X|a'\rangle = \sum_b \langle\alpha|b\rangle\langle b|X|a'\rangle$$

so $\langle\gamma| \doteq \left(\langle\gamma|a^{(1)}\rangle, \ldots\ldots, \langle\gamma|a^{(N)}\rangle\right)$

$= \left(\langle a'|\gamma\rangle^*, \ldots\ldots, \langle a^{(N)}|\gamma\rangle^*\right)$



Finally our inner product is also simple in terms of a row x a column.

The Stern-Gerlach System (i.e. spin 1/2) as an example:

Base ket $|S_z; \pm\rangle = |\pm\rangle$ eigenkets of the S_z operator

$$\mathbb{1} = |+\rangle\langle+| + |-\rangle\langle-|$$

S_z is diagonal here

$$\langle a|S_z|b\rangle = \frac{\hbar}{2}\, \underset{\substack{\uparrow \\ 1}}{b}\, \langle a|b\rangle = \frac{\hbar}{2}\,\delta_{a,b}\, b$$

$a, b = \pm$

$$\Rightarrow \quad S_z = \frac{\hbar}{2}\Big[\,|+\rangle\langle+| - |-\rangle\langle-|\,\Big]$$

In the representation $\quad |S_z;+\rangle = |+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$

$|S_z;-\rangle = |-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$

$$S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Using the ket math: Measurements, observables, and uncertainity relations

<u>Measurement:</u> Before the measurement of observable A the system is generically in some linear combination of eigenkets of A:

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

The result of the measurement is to put the system into one eigenket of A:

$$|\alpha\rangle \xrightarrow[\text{A measurement}]{} |a''\rangle \qquad \underset{\text{measured to be } a''}{A \text{ is then}}$$

The probability for measuring $a' = |\langle a'|\alpha\rangle|^2$

assuming that $|\alpha\rangle$ is normalized.

To determine $|\langle a'|\alpha\rangle|^2$ experimentally we need many measurements of the physical system all in the same quantum state $|\alpha\rangle$ ← Pure ENSEMBLE

Note this postulate agrees w/ our notations of probability

$$|\langle a'|\alpha\rangle|^2 \geq 0 \quad \text{and} \quad \sum_{a'} |\langle a'|\alpha\rangle|^2 = \sum_{a'} \langle a'|\alpha\rangle\langle a'|\alpha\rangle^*$$

$$= \sum_{a'} \langle a'|\alpha\rangle\langle\alpha|a'\rangle = \langle\alpha|\alpha\rangle = 1.$$

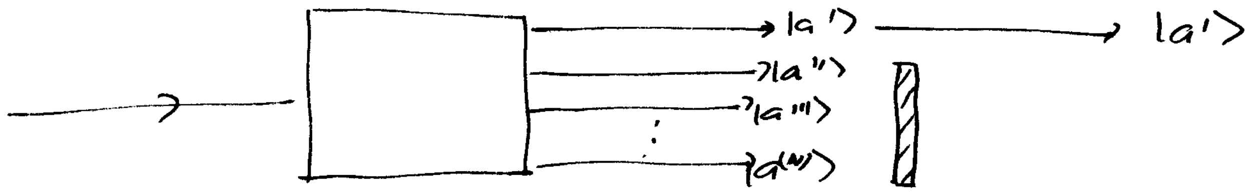
We define the expectation value $\langle A\rangle = \langle\alpha|A|\alpha\rangle$ of A taken wrt the state $|\alpha\rangle$

This makes sense as the average of the measured values:

$$\langle A\rangle = \sum_{a',a''} \langle\alpha|a'\rangle\underbrace{\langle a'|A|a''\rangle}_{=\delta_{a',a''}a'}\langle a''|\alpha\rangle = \sum_{a'} \underset{\text{measured value}}{a'}\ \underset{\text{Prob of measurement}}{|\langle a'|\alpha\rangle|^2}$$

In a more general sense we can define a filter

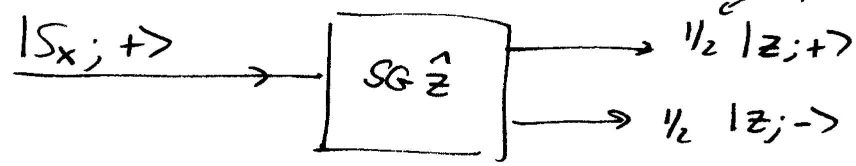
A measurement device that selects only one state of A.



A Stern-Gerlach "A" device with wall $= \Lambda_{a'} = |a'\rangle\langle a'|$

projection operator

Examples from Spin-1/2 systems:



probability

probability.

$\Rightarrow$

$$|x;+\rangle = \frac{1}{\sqrt{2}}\left[|z;+\rangle + e^{i\delta}|z;-\rangle\right]$$

phase angle that is so far unknown.

chose overall phase to be 1.

$|x;-\rangle$ must orthogonal to $|x;+\rangle$ (Why?)

$$|x;-\rangle = \frac{1}{\sqrt{2}}\left[\underset{|+\rangle}{|z;+\rangle} - e^{i\delta}\underset{|-\rangle}{|z;-\rangle}\right]$$

We can now construct the S_x operator (in the S_x basis)

$$S_x = \frac{\hbar}{2}\left[|x;+\rangle\langle x;+| - |x;-\rangle\langle x;-|\right]$$ and in the $|\pm\rangle$

basis of S_z:

$$|x;+\rangle\langle x;+| = \frac{1}{2}\left[|+\rangle + e^{i\delta}|-\rangle\right]\left[\langle +| + e^{-i\delta}\langle -|\right]$$

$$= \frac{1}{2}\left[|+\rangle\langle +| + e^{-i\delta}|+\rangle\langle -| + e^{i\delta}|-\rangle\langle +| + |-\rangle\langle -|\right]$$

$$-|x;-\rangle\langle x;-| = -\frac{1}{2}\left[|+\rangle - e^{i\delta}|-\rangle\right]\left[\langle +| - e^{-i\delta}\langle -|\right]$$

$$= -\frac{1}{2}\left[|+\rangle\langle +| - e^{-i\delta}|+\rangle\langle -| - e^{i\delta}|-\rangle\langle +| + |-\rangle\langle -|\right]$$

$$\Rightarrow \frac{2}{\hbar}S_x = \frac{1}{2}\Big[|+\rangle\langle +| + e^{-i\delta}|+\rangle\langle -| + e^{i\delta}|-\rangle\langle +| + |-\rangle\langle -| - \left(|+\rangle\langle +| - e^{-i\delta}|+\rangle\langle -| - e^{i\delta}|-\rangle\langle +| + |-\rangle\langle -|\right)\Big]$$

$$\frac{2}{\hbar}S_x = e^{-i\delta}|+\rangle\langle -| + e^{i\delta}|-\rangle\langle +|$$

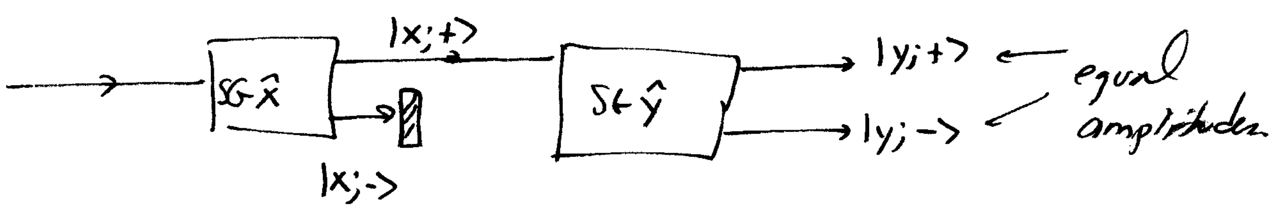
$$\Rightarrow \boxed{S_x = \frac{\hbar}{2}\left[e^{-i\delta}|+\rangle\langle -| + |-\rangle\langle +|e^{i\delta}\right]} \leftarrow \text{Note: this is Hermitian!}$$

We could do the same with S_y:

$$S_y = \frac{\hbar}{2}\left[e^{-i\bar{\delta}}|+\rangle\langle -| + |-\rangle\langle +|e^{i\bar{\delta}}\right]$$

How do we find $\delta, \bar{\delta}$?

If we consider SG experiment of $\hat{x}$ followed by $\hat{y}$



$$\Rightarrow \quad |\langle S_x;+|S_y;+\rangle| = |\langle S_x;+|S_y;-\rangle| = 1/\sqrt{2}$$

so this means:

$$\langle x;+|y;+\rangle = \frac{1}{2}\left[\langle +| + e^{-i\delta}\langle -|\right]\left[|+\rangle + e^{i\bar{\delta}}|-\rangle\right]$$

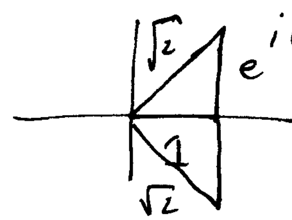
$$= \frac{1}{2}\left[1 + e^{i(\bar{\delta}-\delta)}\right] \quad \text{and}$$

$$\langle x;+|y;-\rangle = \frac{1}{2}\left[1 - e^{i(\bar{\delta}-\delta)}\right]$$

so

$$\frac{1}{2}\left|1 \pm e^{i(\bar{\delta}-\delta)}\right| = \frac{1}{\sqrt{2}}$$

or $\quad e^{i(\bar{\delta}-\delta)} = \pm i$



$e^{i(\bar{\delta}-\delta)} \leftarrow$ must be i or $-i$

$$\Rightarrow \quad \bar{\delta} - \delta = \pm \pi/2$$

People generally take S_x to be real so lets take $\delta = 0$

For a right handed coordinate system we take $\bar{\delta} = \pi/2$ (more on this later)

$$\Rightarrow \quad S_x = \frac{\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]$$

$$S_y = \frac{\hbar}{2}\left[-i\,|+\rangle\langle -| + i\,|-\rangle\langle +|\right]$$

$$S_z = \frac{\hbar}{2}\left[|+\rangle\langle +| - |-\rangle\langle -|\right]$$

a. You can check that the components of $\vec{S}$ obey the commutation relations:

$$[S_i, S_j] = i\,\varepsilon_{ijk}\,\hbar\, S_k$$

antisymmetric tensor w/ $\varepsilon_{123} = +1$.

(Recall $[A,B] = AB - BA$.)

We will see that this is a general consequence of the fact that $\vec{S}$ is an angular momentum observable

b. You can also check that $\{S_i, S_j\} = \frac{1}{2}\hbar\, \delta_{ij}$

where the anti-commutator is defined by $\{A,B\} = AB + BA$.

This is a special property of $\frac{1}{2}\hbar$ angular momentum states

c. We can define the operator $S^2 = S_x^2 + S_y^2 + S_z^2$ — the squared magnitude of the angular momentum. For a $\frac{1}{2}$-spin system:

$$S^2 = \frac{3}{4}\hbar^2\, \mathbb{1}.$$

d. Even though it is not true in general, the following property does ~~not~~ hold generally:

$$[S^2, S_i] = 0 \qquad i = x, y, z.$$

e.

We can define raising and lowering operators:

$$S_\pm = S_x \pm i S_y$$

$$S_+ = \frac{\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +| + |+\rangle\langle -| - |-\rangle\langle +| \right]$$

$$S_+ = \hbar\, |+\rangle\langle -| \quad \leftarrow \text{Non Hermitian}$$

$$S_+ |+\rangle = 0 \qquad S_+ |-\rangle = \hbar\, |+\rangle$$

Similarly $\quad S_- = \hbar\, |-\rangle\langle +|$

Compatible Observables: -

A, B compatible $\Leftrightarrow [A, B] = 0$

Consider two compatible observables A and B

ket space spanned by A. But it is also spanned by B.

We will be able to label all the kets by a set of basis kets

$$|a, b\rangle$$

$\uparrow \uparrow$

eigenvalues of A and B respectively.

* Note we now can admit the idea of degeneracy — more than one state can have the same eigenvalue b

$$B\,|a', b\rangle = b\,|a', b\rangle$$
$$B\,|a'', b\rangle = b\,|a'', b\rangle$$

Generally, we can uniquely label states by a small set of eigenvalues of compatible operators.