A more formal introduction to the vector spaces of Q.M. and the dirac notation

Ket Space: Consider a vector (Hilbert) Space finite or countably infinite (uncountably infinite) dimensions.

A state vector represents a physical state i.e. definite spin orientation of a Ag atom or definite photon polarization.

We call the state vector a ket, $|\alpha\rangle$

$$|\alpha\rangle + |\beta\rangle = |\psi\rangle \qquad \text{addition of kets.}$$

$$c\,|\alpha\rangle = |\alpha\rangle c \qquad \text{multiplication by a c-number}$$

$$c \in \mathbb{C}.$$

$c=0 \Rightarrow$ "null ket"

The states $|\alpha\rangle$ and $c|\alpha\rangle$ represent the same state.

An observable (eg. spin) is represented by an operator in the vector space of states.

$$A|\alpha\rangle = |\beta\rangle \qquad \text{operator acting from the left.}$$

Eigenkets are the eigenvectors of A $\quad |a'\rangle, |a''\rangle, \ldots$

$$A|a'\rangle = \underset{\substack{\uparrow \\ \text{eigenvalue.}}}{a'}|a'\rangle$$

A physical state corresponding to an eigenket is an eigenstate

e.g. spin 1/2 system

$$S_z \mid S_z; \pm\rangle = \pm \hbar/2 \mid S_z; \pm\rangle$$ ← Note: a 2 dimensional space

(dimensionality = # alternatives in a SG type experiment)

Generally, in a N-dimensional space spanned by the N eigenkets of the observable A

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$$ ← eigenkets $|a'\rangle, |a''\rangle, \ldots, |a^{(N)}\rangle$

arbitrary ket

complex #'s

Dual Space: Bra space and inner products

$$\langle\alpha| \xleftrightarrow{DC} |\alpha\rangle \; ; \quad a|\alpha\rangle + b|\beta\rangle \underset{DC}{\longleftrightarrow} a^*\langle\alpha| + b^*\langle\beta|$$

Dual Correspondence

Inner product $\langle\beta|\alpha\rangle = \langle\beta| \cdot |\alpha\rangle$

bra (c) ket

Postulates

(I) ~~Note~~ $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$

$\Rightarrow$ $\langle\alpha|\alpha\rangle$ is real let $|\alpha\rangle = |\beta\rangle$ above.

(II) $\langle\alpha|\alpha\rangle \geq 0$ and $= 0$ only for the null ket.

positive definite metric.

* Two kets are orthogonal iff $\langle\alpha|\beta\rangle = 0$

* A non null ket can be normalized $|\alpha\rangle$

$$|\tilde{\alpha}\rangle = \frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}|\alpha\rangle$$

normalized ket

Norm of $|\alpha\rangle$.

Operators:

* Act from the left on kets $X|\alpha\rangle$ and generate a new ket. They can operate on a bra to the left $\langle\alpha|X$
* $X = Y$ Two operators are equal if

$X|\alpha\rangle = Y|\alpha\rangle$ for an arbitrary ket $|\alpha\rangle$.

$$\boxed{\begin{array}{c}\underline{\text{Definition of the Hermitian adjoint}} \\ X|\alpha\rangle \xleftrightarrow[DC]{} \langle\alpha|X^{+}\end{array}}$$

X is Hermitian iff $X = X^{+}$

Generally operator multiplication is noncommutative i.e.

$$XY \neq YX$$

but it is associative: $X(YZ) = (XY)Z = XYZ$

Note that

$$XY|\alpha\rangle = X(Y|\alpha\rangle) \xleftrightarrow[DC]{} (\langle\alpha|Y^{+})X^{+} = \langle\alpha|Y^{+}X^{+}$$

So $(XY)^+ = Y^+ X^+$

So we can interpret: $|\alpha\rangle$ $\langle\beta|$ $\langle\beta|\alpha\rangle$ $X|\alpha\rangle$ $\langle\alpha|X$ and XY. We can also form an outer product

$|\alpha\rangle\langle\beta|$ ← This is an operator!

Note that $|\alpha\rangle|\beta\rangle$ and $\langle\alpha|\langle\beta|$ don't have a meaning if $|\alpha\rangle, |\beta\rangle$ are in the same vector space.

The Associative Axiom

Consider $\underbrace{\big(|\alpha\rangle\langle\beta|\big)}_{\text{operator}} \cdot \underbrace{|\gamma\rangle}_{\text{ket}} = \underbrace{|\alpha\rangle}_{\text{ket}} \underbrace{\langle\beta|\gamma\rangle}_{\text{c-number}}$

or operator × ket = ket. as required!

Example of the Associative Axiom in action:

I) Say $X = |\beta\rangle\langle\alpha|$ show that $X^+ = |\alpha\rangle\langle\beta|$

$$X|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle$$

$$\text{DC} \updownarrow \qquad\qquad \updownarrow \text{DC}$$

$$\left.\begin{aligned}\langle\gamma|X^+ &= \langle\alpha|\gamma\rangle^* \langle\beta| \\ &= \langle\gamma|\alpha\rangle\langle\beta|\end{aligned}\right\} \Rightarrow \quad X^+ = |\alpha\rangle\langle\beta|$$

II) Consider $\big(\langle\beta|\big)\big(X|\alpha\rangle\big) = \big(\langle\beta|X\big)\cdot|\alpha\rangle$

Both can be written as: $\langle\beta|X|\alpha\rangle$

Note also that

$$\langle\beta|X|\alpha\rangle = (\langle\beta|)\cdot X|\alpha\rangle \qquad \text{since } \langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$$

DC

$$= \left\{ (\langle\alpha|X^\dagger)\cdot|\beta\rangle \right\}^*$$

$$= \langle\alpha|X^\dagger|\beta\rangle^*$$

For the special case of a Hermitian operator X

$$\langle\alpha|X|\beta\rangle^* = \langle\beta|X|\alpha\rangle$$

Base kets and Matrix Representations

A = Hermitian operator

<u>Thm.</u> The eigenvalues of Hermitian operator A are real; the eigenkets of A corresponding to different eigenvalues are orthogonal

<u>Pf</u>:

$$A|a'\rangle = a'|a'\rangle \quad \text{and} \quad \langle a''|A = a''^*\langle a''|$$

Because $A|a''\rangle \overset{}{\underset{DC}{\longleftrightarrow}} \langle a''|A^\dagger = \langle a''|A$

so

$$\langle a''|A|a'\rangle = a'\langle a''|a'\rangle = a''^*\langle a''|a'\rangle$$

$$\Rightarrow \quad (a' - a''^*)\langle a''|a'\rangle = 0$$

if $a' = a''$ $\Rightarrow$ $a' - a'^* = 0$ $\Rightarrow$ $a' = a'^*$ $\therefore$ a' is real

if a', a'' are different: $a''^* = a''$ so

$$(a' - a'')\langle a''|a'\rangle = 0 \text{ and } a' \neq a'' \text{ so}$$

$\langle a''|a'\rangle = 0 \Rightarrow |a''\rangle$ and $|a'\rangle$ are orthogonal.

It is typical done to normalize $|a'\rangle$ so

$\{|a'\rangle\}$ is orthonormal. i.e. $\langle a'|a''\rangle = \delta_{a',a''}$

Since the physical space is spanned by the eigenkets of A, we have an orthonormal basis

Eigenkets as base kets.

Normalized eigenkets of A form a complete orthonormal set. So, we can expand any ket in the ket space spanned by the eigenkets of A as:

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$$

$$\langle a''|\alpha\rangle = \sum_{a'} \langle a''|a'\rangle c_{a'} = c_{a''} ; \quad \boxed{c_{a''} = \langle a''|\alpha\rangle}$$

Expansion coefficients

From the associative axiom:

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle = \left(\sum_{a'} |a'\rangle\langle a'|\right)\cdot|\alpha\rangle$$

$$\Rightarrow \quad \mathbb{1} = \sum_{a'} |a'\rangle\langle a'| \quad \text{Completeness or closure.}$$

Analogous to: $\hat{e}_x\hat{e}_x + \hat{e}_y\hat{e}_y + \hat{e}_z\hat{e}_z = \overleftrightarrow{\mathbb{1}}$ in Cartesian Space.

We can now define a <u>projection</u> <u>operator</u>:

$$\Lambda_{a'} = |a'\rangle\langle a'| \quad ; \quad \sum_{a'} \Lambda_{a'} = \mathbb{1}.$$

Note that our polaroid filter acts like a projection operator.

$$|\psi\rangle \longrightarrow [\,] \quad |x\rangle\langle x|\psi\rangle = \left(|x\rangle\langle x|\right)\cdot|\psi\rangle$$

x-filter

x-polarized photon

probability amplitude of getting through.

Matrix Representations of Operators.

For any operator X we can use closure to write

$\{|a'\rangle, \dots, |a^{(N)}\rangle\}$

$$X = \sum_{a', a''} |a''\rangle\langle a''|\, X\, |a'\rangle\langle a'|$$

N^2 c-numbers.

We can write them in matrix form:

$$\langle a''|X|a'\rangle$$

row

column

$$X \doteq \begin{pmatrix} \langle a^1|X|a^1\rangle & \langle a^1|X|a^2\rangle & \langle a^1|X|a^3\rangle & \cdots \\ \langle a^2|X|a^1\rangle & \langle a^2|X|a^2\rangle & \langle a^2|X|a^3\rangle & \\ \langle a^3|X|a^1\rangle & & & \end{pmatrix}$$

X is represented by...