Functions of a Complex Variable

Why do we care?

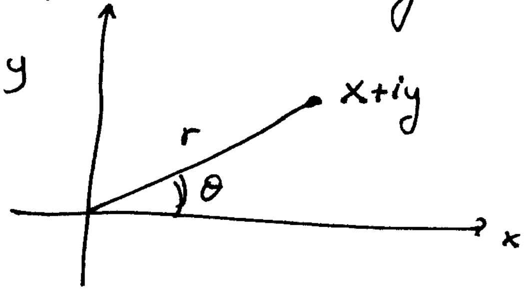
- electrostatics, hydrodynamics etc. solution of the Laplace equation. We will see how these are related to the theory of complex functions
  - Solving second order differential equations.
  - Integral Transforms.

Complex Algebra

$i = \sqrt{-1}$. A complex number $z = x + iy$

Actually isomorphic to matrices of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. This is just a side note, but it has interesting applications in terms of the algebra of Pauli matrices.

The complex Plane (Argand diagram)



$\theta \Rightarrow$ called the "argument"

$r \Rightarrow$ called the "modulus"

$z = r\cos\theta + i r\sin\theta = re^{i\theta}$ $\Leftarrow$ Polar form of complex numbers.

Note that : 1) $|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

2) $|z_1 z_2| = |z_1| \, |z_2|$

3) $\arg(z_1 \bullet z_2) = \arg(z_1) + \arg(z_2)$

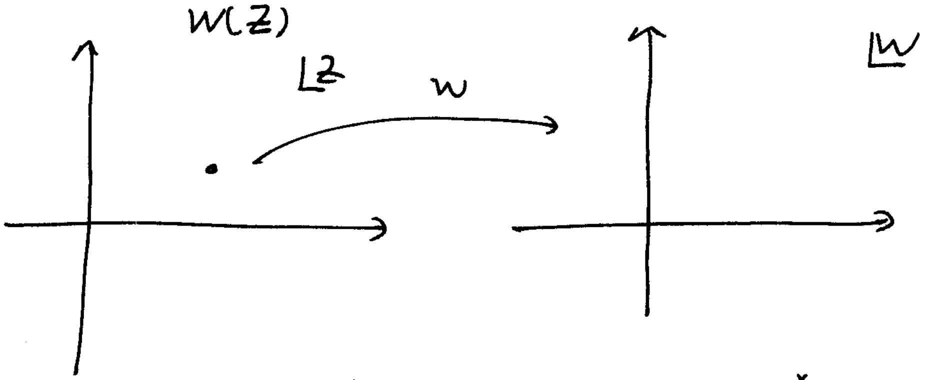
Functions of a complex variable: $w(z) = u(x,y) + i v(x,y)$

Example $f(z) = z^2 = x^2 - y^2 + 2i\,xy$; where $z = x+iy$

Real and Imaginary parts of a function

$\mathrm{Re}\, w(z) = u(x,y)$ $\quad \mathrm{Im}\, w(z) = v(x,y)$ ← Note that both of these are real.

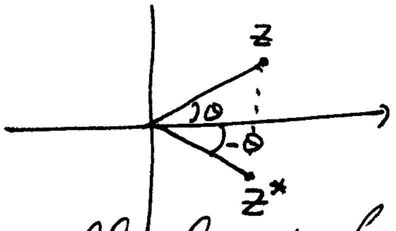
Note that a complex function is a mapping from the complex plane to itself.



Complex Conjugation: $z = x+iy$ $\quad z^* = x - iy$

$zz^* = |z|^2$ check this!

and in the complex plane



We can analytically continue all elementary functions to the complex plane

e.g. A. $e^{in\theta} = \cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n$

(De Moivre's theorem)

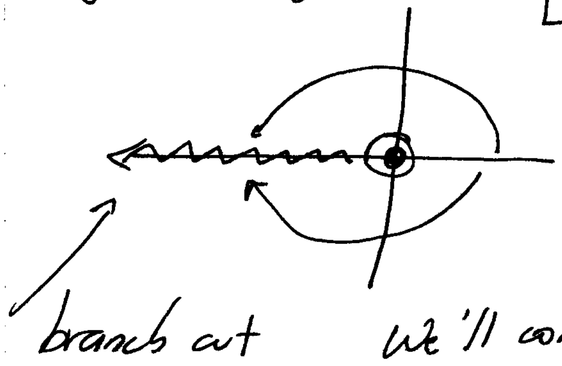
Watch out for multiple valued functions! The most extreme example among well-known elementary functions:

$\log z = \log(re^{i\theta}) = \log r + i\theta$ but we can add $2n\pi$ to the argument and have the same z!

$\log z = \log(r) + i(\theta + 2n\pi)$, $n \in \mathbb{Z}$.

Lz ← Infinitely many values.

set $n=0$ (principle value) and take $-\pi < \theta < \pi$ Traditional choice.



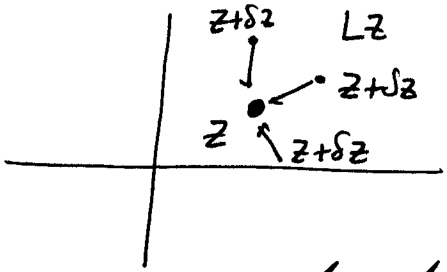
We'll come back to this point later.

Cauchy-Riemann Conditions and the derivatives of complex functions.

Can we define $\frac{df}{dz}$ for $f(z)$?

How about: $\frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$?

But what direction can we δz to zero.



Ans: we have to be sure that the direction doesn't matter!

Say $\delta z = \delta x + i\,\delta y$ and $f = u + iv$

$\frac{\delta f}{\delta z} = \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$ say $\delta y = 0 \Rightarrow$

$\lim \frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ or say $\delta x = 0 \Rightarrow$

$\lim \frac{\delta f}{\delta z} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$\Rightarrow$ comparing these we see we need

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

If the derivative is well-defined, these Cauchy-Riemann conditions hold.

Conversely, if the CR conditions hold, $\frac{df}{dz}$ exists.

Consider

$$\delta f = (\partial_x u + i\partial_x v)\delta x + (\partial_y u + i\partial_y v)\delta y$$

$$\Rightarrow \frac{\delta f}{\delta z} = \frac{(\partial_x u + i\partial_x v)\delta x + (\partial_y u + i\partial_y v)\delta y}{\delta x + i\delta y}$$

$$\text{or} \quad \frac{\delta f}{\delta z} = \frac{\partial_x u + i\partial_x v + \frac{\delta y}{\delta x}\left[\partial_y u + i\partial_y v\right]}{1 + i\frac{\delta y}{\delta x}}$$

Apply the CR conditions

$$\partial_y u + i\partial_y v = -\partial_x v + i\partial_x u = i(\partial_x u + i\partial_x v)$$

$$\Rightarrow \frac{\delta f}{\delta z} = \frac{(\partial_x u + i\partial_x v)\left[1 + i\,\delta y/\delta x\right]}{1 + i\,\delta y/\delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

independent of $\delta y/\delta x$ ← how we approach z.

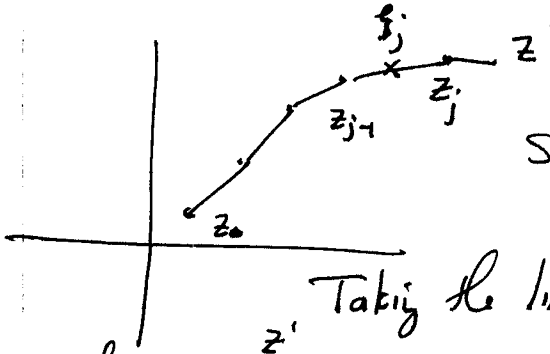
Prove that curves of u = const. are everywhere orthogonal to curves of v = const. if $u+iv$ is a differentiable function.

Ans: Use the CR conditions.

If $f(z)$ is differentiable at $z = z_0$ and in some region about this point $f(z)$ is called analytic at z_0.

Is $f(z) = z^2$ analytic? Is $f(z) = z^*$, is it continuous?

Cauchy's Integral thm:



$$S_n = \sum_{j=1}^{n} f(\xi_j)\,(z_j - z_{j-1})$$

Taking the limit $|z_j - z_{j-1}| \to 0$ and $n \to \infty$ we get

$$\int_{z}^{z'} dz\, f(z)$$ along the path specified

Notice that a complex integral is a sum of real integrals.

Cauchy's Integral thm: if C is a closed contour and f(z) is an analytic function w/ continuous partial derivatives throughout a simply connected region R. If C is contained in R

$$\oint_C f(z)dz = 0$$ ← Like conservative forces!

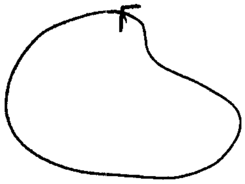
* We don't need to require the continuity of the partial derivatives but it makes the thm easier to prove and is usually true in physical problems. If you want to see a proof under less restrictive conditions, it is sketched in Arfken – see the Goursat proof.

$f = u + iv$ and $dz = dx + i\,dy$

$$\Longrightarrow \oint f(z)\,dz = \oint (u\,dx - v\,dy) + i\int v\,dx + u\,dv$$

Use Stokes Thm: with $u = V_x$ and $-v = V_y$ for some vector field $\vec{V} = V_x\,\hat{x} + V_y\,\hat{y}$

$$\oint_C (V_x\,dx + V_y\,dy) = \int \underset{\overset{\uparrow}{(\vec{\nabla}\times\vec{V})_z}}{\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)} dx\,dy$$



so

$$\oint_C (u\,dx - v\,dy) = \int dx\,dy\,\underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\text{zero by the CR condition}}$$

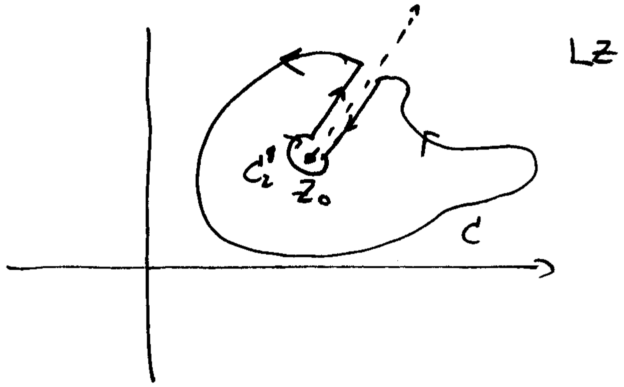
$\partial_x v = -\partial_y u$

You can do the same for the second integral.

Now we can prove that

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0)$$ w/ the same condition on C and R, and $f(z)$.

f(z) is analytic but the integrand is $\underline{\underline{\text{not}}}$ at $z=z_0$ where the function and its derivatives diverge.



on the contour shown above $\oint \frac{f(z)}{z-z_0}dz = 0$ so

$$\oint_C dz\,\frac{f(z)}{z-z_0} = -\oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

i.e. we can integrate the wrong way around In other words,

$$\oint dz\,\frac{f(z)}{z-z_0} = \oint_{\bar{C}_2} dz\,\frac{f(z)}{z-z_0}$$

counter clockwise

on this contour $\bar{C}_2$:

$z = z_0 + re^{i\theta}$

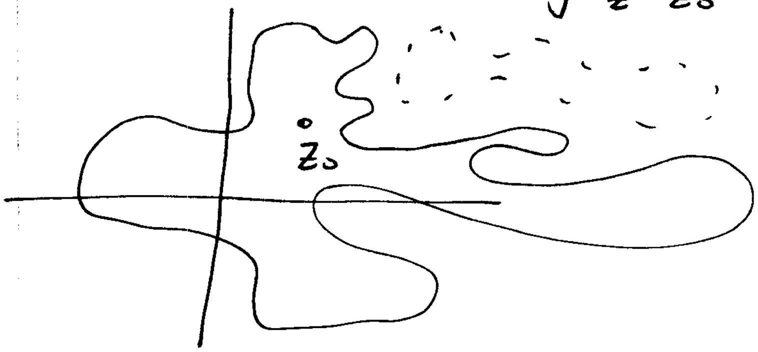
$dz = rie^{i\theta}d\theta$

so
$$\oint dz\,\frac{f(z)}{z-z_0} = \lim_{r\to 0}\int \frac{f(z_0+re^{i\theta})\,ri\,d\theta\,e^{i\theta}}{re^{i\theta}}$$

$$= f(z_0)\int_0^{2\pi} i\,d\theta = 2\pi i\, f(z_0)$$

It is worthwhile to pause and consider this surprising result:

For an analytic function f, $\oint \frac{dz\, f(z)}{z-z_0}\frac{1}{2\pi i} = \begin{cases} f(z_0) & z_0 \text{ interior} - \\ 0 & z_0 \text{ exterior} \dots \end{cases}$



Two contours: — $\Rightarrow f(z_0)$

···· $\Rightarrow 0$.

We can not only use the integral to give the value of the analytic function at z_0, but we can also compute its derivatives there:

Consider:

$$I = \oint \frac{f(z)}{(z-z_0)^2}\, dz$$

Doing the same computation we get

enclosed $z_0 \longleftrightarrow C$ $\qquad z = z_0 + re^{i\theta}$

$$I = \lim_{r\to 0} \oint \frac{f(z_0+re^{i\theta})\, ir\, d\theta\, e^{i\theta}}{(re^{i\theta})^2}$$

Note: This requires the validity of the Taylor expansion

$$I = \lim_{r\to 0} \oint \frac{f(z_0)\, i\, d\theta}{re^{i\theta}} + \oint \frac{\frac{df}{dz}\Big|_{z_0} re^{i\theta}\, ire^{i\theta}\, d\theta}{r^2 e^{2i\theta}} + \text{terms that go to zero as } r\to 0$$

$$I = \lim_{r\to 0}\left[\frac{i}{r} f(z_0) \overset{0}{\cancel{\int_0^{2\pi} e^{-i\theta}\, d\theta}} + i \frac{df}{dz}\Big|_{z_0} \int_0^{2\pi} d\theta \right]$$

$$I = 2\pi i\, f'(z_0)$$

Can also get this result from $\frac{f(z+\delta z)-f(z)}{\delta z}$ and direct calculation

In general you should be able to show that

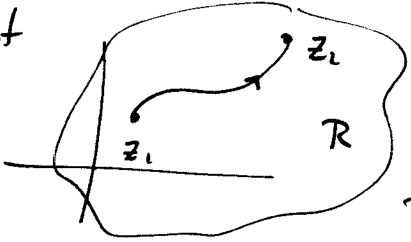
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

⟹ Note that an analytic function must have well-defined derivatives to all orders. After all, we just computed them.

Morera's Thm: Converse of Cauchy's Thm

If $f(z)$ is continuous in a simply connected domain R and $\oint_C f(z)dz = 0$ for every closed contour C in R, then f is analytic in R.

Pf: Integrate from z_1 to z_2 — note that this is path independent



$$\underbrace{F(z_2) - F(z_1)}_{\text{Depends only on the endpoints}} = \int_{z_1}^{z_2} dz\, f(z)$$

Now

$$\frac{F(z_2)-F(z_1)}{z_2-z_1} - f(z_1) = \frac{\int_{z_1}^{z_2} dt\,[f(t)-f(z_1)]}{z_2-z_1}$$

Now take the limit $\lim_{z_2\to z_1} \frac{\int_{z_1}^{z_2} dt\,[f(t)-f(z_1)]}{z_2-z_1} = 0$ Why?

By the mean value thm: $\int_{z_1}^{z_2} dt\, f(t) = f(z')(z_2 - z_1)$

where $z_1 < z' < z_2$ along the integration path.

Since $f(z)$ is continuous $f(z') \rightarrow f(z_1)$ as $z_2 \rightarrow z_1$

Therefore $$\underbrace{\lim_{z_2 \to z_1} \frac{F(z_2) - F(z_1)}{z_2 - z_1}}_{F'(z_1)} = f(z_1)$$

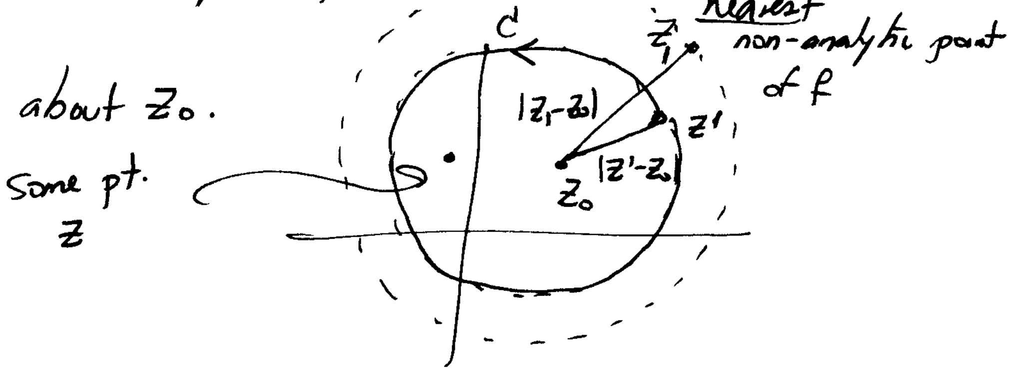
so $F'(z)$ exists at z_1 and is equal to $f(z_1)$ $\Rightarrow$ $F(z)$ is analytic and $F'(z) = f(z)$ in R.

Then F'' exists and is equal to $f'(z)$ — f is analytic in R.

Laurent Expansion:

How to do a Taylor expansion for a complex function

Expand $f(z)$ about z_0.



So f is analytic on and within C

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z}$$ where z is on the interior of C

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)}$$

$$= \frac{1}{2\pi i}\oint \frac{f(z')\,dz'}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]}$$

We note that for $|t|<1$

$$\frac{1}{1-t} = 1+t+t^2+t^3+\cdots = \sum_{n=0}^{\infty} t^n$$

since $\frac{|z-z_0|}{|z'-z_0|}<1$ we have

$$f(z) = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} \frac{\overbrace{(z-z_0)^n}^{\text{We are not integrating over this!}} f(z')\,dz'}{(z'-z_0)^{n+1}}$$ or

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \frac{f^{(n)}(z)}{n!}$$

Analytic Continuation

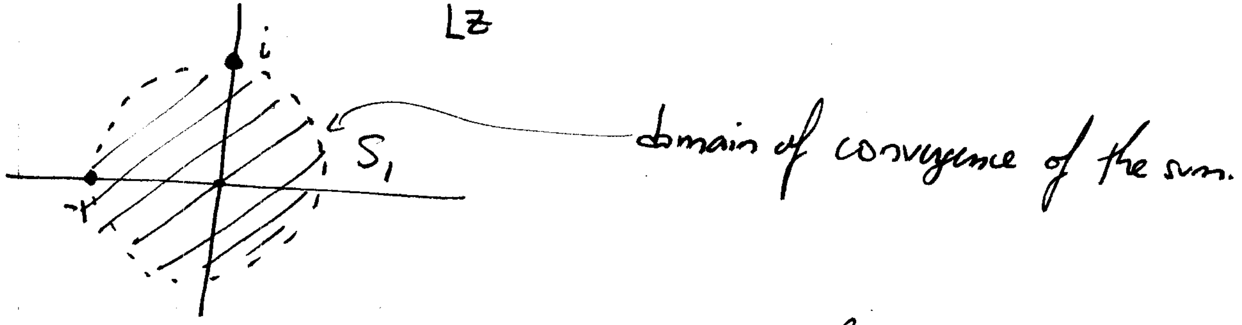
We assumed f(z) had a nonanalytic point that was isolated like

$$f(z) = \frac{1}{z+1}$$

Now with our expansion (or just the binomial thm)

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n z^n$$

and we know that this is convergent for $|z| < 1$



But we know that f(z) is well-defined and analytic elsewhere in the complex plane.

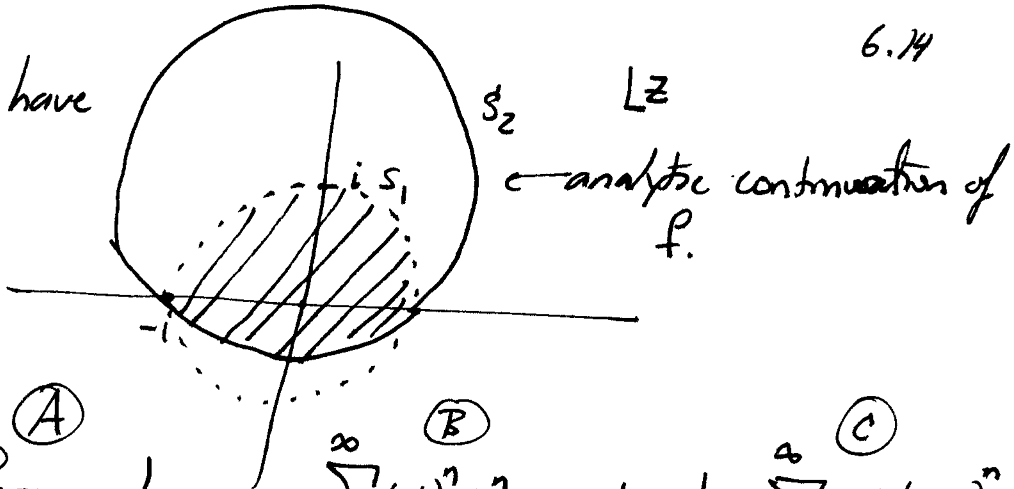
e.g. we can expand f about i

$$f(z) = \frac{1}{1+z} = \frac{1}{1+i+(z-i)} = \frac{1}{1+i}\left[1 + \frac{z-i}{1+i}\right)^{-1}$$

$$f(z) = \frac{1}{1+i}\left[1 - \frac{z-i}{1+i} + \left(\frac{z-i}{1+i}\right)^2 - \cdots \right]$$

This is convergent for $|z-i| < |1+i| = \sqrt{2}$

Now we have



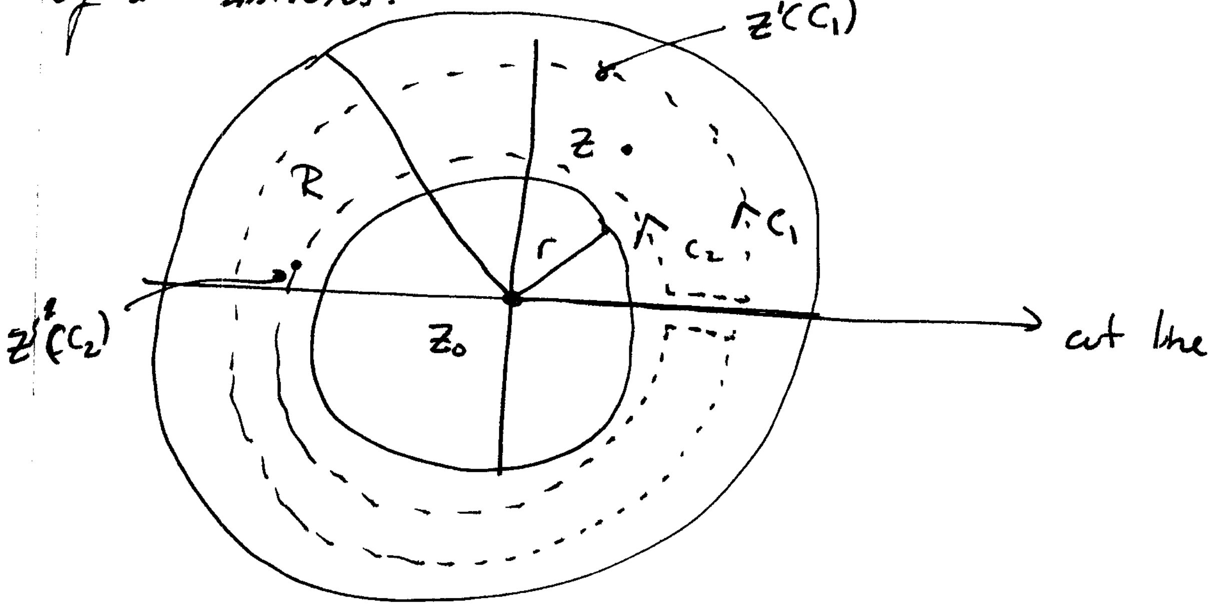
So $f(z) = \overset{Ⓐ}{\frac{1}{1+z}}$, $\overset{Ⓑ}{\sum_{n=0}^{\infty}(-1)^n z^n}$, and $\overset{Ⓒ}{\frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{1+i}\right)^n}$

are three different representations of the same function.

Ⓑ is a Maclaurin series, Ⓒ is a Taylor expansion about $z=i$, and Ⓐ is a one-term Laurent series.

What is a Laurent Series?

We may find a function that is analytic on the interior of an annulus:



$$f(z) = \frac{1}{2\pi i}\underset{C_1\ \textcircled{A}}{\oint} \frac{f(z')dz'}{z'-z} \underset{\uparrow}{-} \frac{1}{2\pi i}\underset{C_2\ \textcircled{B}}{\oint} \frac{f(z')dz'}{z'-z}$$

note the minus sign because we flipped the direction of integration on the inner loop.

Note also: $|z'-z_0|_{C_1} > |z-z_0| > |z'-z_0|_{C_2}$ ← see the picture on 6.14

On C_1 we can expand the integrand:

$$z'-z = (z'-z_0) - (z-z_0)$$

$$\frac{1}{z'-z} = \frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{z'-z_0}\,\frac{1}{1-\frac{z-z_0}{z'-z_0}}$$

convergent for all points within the larger circle R

$$= \frac{1}{z'-z_0}\sum_{n=0}^{\infty}\left[\frac{(z-z_0)}{(z'-z_0)}\right]^n$$

so the first integral becomes:

$$A = \frac{1}{2\pi i}\oint_{C_1}\sum_{n=0}^{\infty}(z-z_0)^n\frac{dz'f(z')}{(z'-z_0)^{n+1}} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{dz'f(z')}{(z'-z_0)^{n+1}}$$

and the second integral becomes:

$$B = +\frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\oint dz' f(z')(z'-z_0)^{n-1}$$

← convergent for all points outside r

We can write these two series together as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$ where C is *any* (counterclockwise) contour in the annular region $r<|z|<R$

One example for now:

$$f(z) = \frac{1}{z(z-1)}$$. We can take $z_0 = 0$ and $r=0$, $R=1$

$$a_n = \frac{1}{2\pi i}\oint \frac{dz'}{z'^{2+n}(z'-1)} = -\frac{1}{2\pi i}\oint \sum_{m=0}^{\infty} z'^m \frac{1}{z'^{2+n}}$$

$\uparrow$

contour w/ $0<|z'|<1$

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint \frac{dz'}{z'^{n+2-m}} = -\frac{1}{2\pi i}\, 2\pi i \sum_{m=0}^{\infty} \delta_{n+2-m,\,1}$$

$$\Rightarrow \quad a_n = \begin{cases} 0, & n<-1 \\ -1, & n \geq -1 \end{cases}$$

$$\Rightarrow \quad \frac{1}{z(z-1)} = -\frac{1}{z} - 1 - z - z^2 - z^3 - \dots$$

We could have done this the easy way as well...

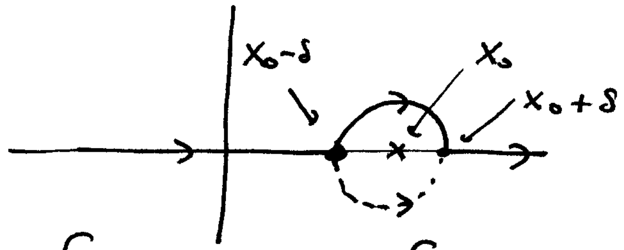
$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$= -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$$

Cauchy Principle Value.

We now know how to handle integrals in the complex plane we the (simple) poles lie away from the contour of integration.

Now we will consider a 1st order pole <u>on</u> the contour of integration. How can we give that some meaning? We can deform the contour to miss the pole:



top → $\pi i\, a$
botton → $-i\pi\, a$

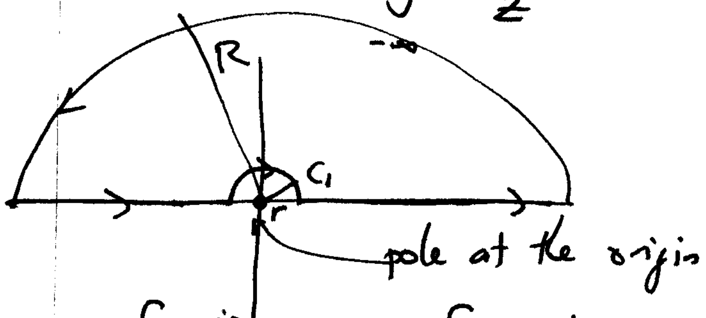
$$\int \frac{f(x)\,dx}{x - x_0} = \mathcal{P} \int \frac{f(x)}{x - x_0}\,dx \pm i\pi f(x_0)$$

↑

Cauchy Principle Value

Example $I = \int_0^\infty \frac{\sin x}{x}\,dx$ (Single Slit Diffraction)

$$2I = \cancel{2}\ \mathrm{Im} \int_{-\infty}^{\infty} \frac{e^{+iz}}{z}\,dz$$



$$0 = \oint \frac{e^{iz}}{z}\,dz = \int_{C_1} \frac{e^{iz}dz}{z} + P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$$

$$\shortparallel$$

$$-i\pi e^{i0} = -i\pi$$

$$\Rightarrow \quad P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = +i\pi \quad \text{or}$$

$$2I = \cancel{2}\,\mathrm{Im} \int \frac{e^{iz}}{z}\,dz = \pi \quad \Rightarrow \quad I = \pi/2.$$

or $\boxed{\int_{-\infty}^{\infty} \frac{dx \sin x}{x} = \pi/2}$

The Method of Steepest Descents.

We can now look at the integral

$$I(s) = \int_c \underset{\substack{\uparrow \\ \text{analytic functions}}}{g(z)}\, e^{s \overset{\uparrow}{f(z)}} dz \quad \text{where} \quad s >> 1$$

we can take s to be real for now.

Re f(z) → ∞ when we approach the limits of integration along contour C

The integral should be dominated by the region where f(z) takes on its maximum value.

$f(z) = u(x,y) + i v(x,y)$; $z = x + iy$

$$I(s) = \int_c g(z)\, \underset{\substack{\uparrow}}{e^{su}}\, e^{isv} dz$$

maximum where $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \Rightarrow$

$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ by the Cauchy Riemann conditions.

i.e. $\frac{df}{dz} = 0$

But is the point where $\frac{df}{dz} = 0$ really a max or min of u, v?

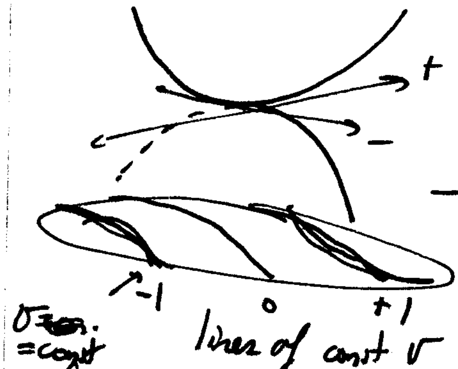
Ans: No! why?

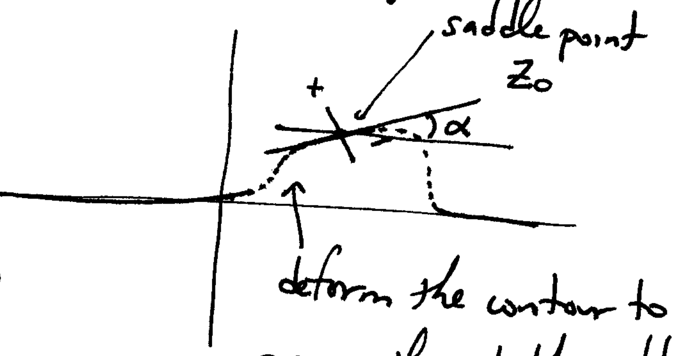
Note curves of v=const. and tangent to $\vec{\nabla} u$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}=0$$

↑ by the CR condition

so u satisfies the Laplace equation in 2d: $\nabla^2 u=0$ and its extrema must be saddle points!





deform the contour to pass through the saddle in the descending direction. Then

$$I(s)\cong e^{isv_0}\int_c g(z)e^{su(x,y)}dz$$

← and expand here about the ~~max~~ saddle in the descending direction

$$f(z)=f(z_0)+\frac{1}{2}(z-z_0)^2\frac{d^2f}{dz^2}\Big|_{z=z_0}+\dots$$

real since v=const

$$f(z)-f(z_0)=\tfrac{1}{2}(z-z_0)^2f''(z_0)=-\frac{1}{2s}t^2$$

defining the new variable t, which is real.

$$z-z_0=\delta e^{i\alpha}\Rightarrow$$

← constant phase

$$t^2=\delta^2e^{2i\alpha}(-sf''(z_0))\;;\quad t=\pm\delta\,|sf''(z_0)|^{1/2}$$

pick the phase so that $\mathrm{Im}[f(z)-f(z_0)]=0$ and $\mathrm{Re}[f(z)-f(z_0)]<0$

Now,

$$I(s) \cong g(z_0)\, e^{s f(z_0)} \int_{-\infty}^{\infty} e^{-t^2/2} \frac{dz}{dt}\, dt$$

$\nearrow$ extend the limits of integration

~~[illegible]~~ $$\frac{dz}{dt} = \left(\frac{dt}{dz}\right)^{-1} = \left(\frac{dt}{d\delta}\frac{d\delta}{dz}\right)^{-1} = |s f''(z_0)|^{-1/2} e^{i\alpha}$$

So

~~$I(s) \cong$ [illegible] $g(z_0)\, e$~~

$$I(s) \cong \frac{\sqrt{2\pi}\, g(z_0)\, e^{s f(z_0)}\, e^{i\alpha}}{|s f''(z_0)|^{1/2}}$$

Example

$$s! = \int_0^\infty x^s e^{-x} dx = s^{s+1} \int_0^\infty e^{-sz} z^s dz$$

Using:
$x = zs$
$dx = s\, dz$

$$= s^{s+1} \int_0^\infty e^{+s(\log z - z)} dz$$

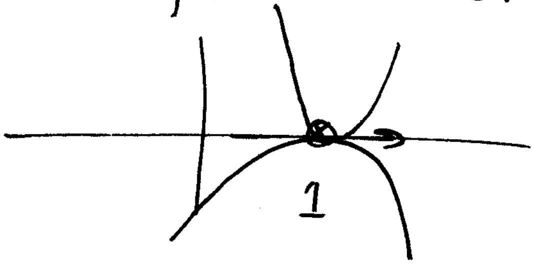
so $f(z) = \log z - z$ $\qquad \frac{df}{dz} = 0 \Rightarrow \frac{1}{z} - 1 = 0$

$\Rightarrow z = 1 \qquad z - 1 = \delta e^{i\alpha}$

And $f(z) = \log(1+\delta e^{i\alpha}) - (1+\delta e^{i\alpha})$

$$= \delta e^{i\alpha} \cancel{+} -\tfrac{1}{2}\delta^2 e^{2i\alpha} - 1 - \delta e^{i\alpha} = -1 - \tfrac{1}{2}\delta^2 e^{2i\alpha}$$

So we pick the contour to follow the real axis ($\alpha=0$)



so w/ $\alpha = 0$ we get

$$s! = \frac{\sqrt{2\pi}\, s^{s+1} e^{-s}}{\left|s \frac{-1}{(+1)^2}\right|^{1/2}} = \underbrace{\sqrt{2\pi s}}_{\text{1st term of the Stirling expansion}}\, s^s e^{-s}$$