Differential Equations

1. First order equations:

$$\frac{dy}{dx} = f(x,y) = -\frac{P(x,y)}{Q(x,y)}$$ ← standard notation

only a 1st derivative ⟹ <u>first order</u>
<u>ordinary</u> ⟺ total derivatives, not partial derivatives

A couple of examples

$$\frac{dy}{dx} = -\frac{P(x)}{Q(y)}$$ ← Separable differential equation

This form is particularly simple to solve:

$$Q(y)dy + P(x)dx = 0$$ Integrate from (x_0, y_0) to (x, y)

$$\int_{y_0}^{y} Q(y)dy + \int_{x_0}^{x} P(x)dx = 0$$

We can throw away these constants and just add a constant of integration to be evaluated some other way.

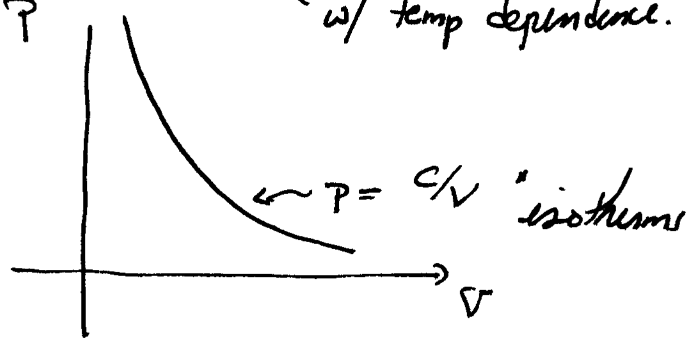
e.g. Boyle's gas law $\frac{dV}{dp} = -V/p$ so

$$\frac{dV}{V} = -\frac{dp}{p} \Rightarrow \log V = -\log p + C$$

or $\log(PV) = c \leftarrow$ const.

$\Rightarrow \quad pV = \text{const.} \leftarrow$ Note for an ideal gas this is just Nk_BT

w/ temp dependence.



There is another type of 1st order differential equation that you know how to solve already

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$ <u>Exact Differentials</u>

i.e. there is some function $\varphi \ni \begin{cases} \partial_x \varphi = P \\ \partial_y \varphi = Q \end{cases}$

Then we have $d\varphi = 0$ so the solutions are:

$$\varphi(x,y) = \text{const.}$$

Note: This just reproduces potential theory that we studied in chapter 1.

In general you can always find an <u>integrating factor</u> $\alpha(x,y)$ so that:

$\alpha(x,y)\,P(x,y)\,dx + \alpha(x,y)\,Q(x,y)\,dy = 0$ and now the LHS is a total differential.

In general, this is tough, but there are simple and commonly found cases where you can find a general prescription for the integrating factor.

Linear 1st order Differential Equations

$$\frac{dy}{dx} + p(x)y = q(x)$$ arbitrary functions of the independent variable.

Linear in the dependent variable $y = y(x)$

we want $\alpha(x)$ such that

$$\alpha y' + \alpha p y = \alpha q \quad \text{and} \quad \alpha y' + \alpha p y = \frac{d}{dx}[\alpha y]$$

so $\alpha p = \alpha' \Rightarrow \alpha = \exp\left[\int^x p(x')dx'\right]$

irrelevant constant.

so we then have:

$$\frac{d}{dx}[\alpha y] = \alpha q \quad \text{or}$$

$$\alpha y = \int^x \alpha(x')q(x')dx' + c$$

$$y(x) = \frac{c}{\alpha(x)} + \frac{1}{\alpha(x)}\int^x \alpha(x')q(x')dx' \; ; \quad \alpha(x) = e^{\int^x p(x')dx'}$$

A simple example: a ball falling in a viscous liquid.

$$m\frac{dv}{dt} = -mg - \xi v$$

Newton's 2^{nd} Law with a drag force proportional to velocity (low Reynolds #)

define $\bar{\xi} = \xi/m$

$$\frac{dv}{dt} + \bar{\xi} v = g \qquad \alpha = e^{\bar{\xi} t}$$

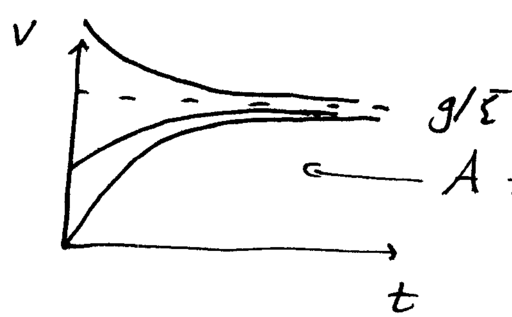
$$e^{\bar{\xi} t}\dot{v} + \bar{\xi} e^{\bar{\xi} t} v = g e^{\bar{\xi} t}$$

multiplied by the integrating factor.

$$\frac{d}{dt}\left[v(t) e^{\bar{\xi} t}\right] = g e^{\bar{\xi} t}$$

$$v(t) e^{\bar{\xi} t} - v(0) = \int_0^t dt'\, g e^{\bar{\xi} t'} = \frac{g}{\bar{\xi}}\left(e^{\bar{\xi} t} - 1\right)$$

$$v(t) = \underbrace{v(0)}_{\text{initial condition}} e^{-\bar{\xi} t} + \underbrace{\frac{g}{\bar{\xi}}}_{\text{Terminal velocity}}\left(1 - e^{-\bar{\xi} t}\right)$$



A family of solutions with different initial conditions.

2^{nd} order Differential Eqns - Linear

Two independent solutions ← more on that later.

A. We constant coefficients Homogeneous case

$$y'' + ay' + by = 0 \leftarrow \text{homogeneous}$$

Linear.

This particularly simple problem has a nice, closed form solution:

Ansatz $y \sim e^{\alpha x} \Rightarrow$

$$\alpha^2 + a\alpha + b = 0 \Rightarrow \alpha = \frac{-a \pm \overbrace{\sqrt{a^2 - 4b}}^{\Delta}}{2}$$

Two cases 1) α real when $a^2 > 4b$

2) α complex when $a^2 < 4b$

1) $y \sim A e^{\left(-\frac{a}{2} + \frac{\Delta}{2}\right)x} + B e^{\left(-\frac{a}{2} - \Delta/2\right)x}$

2) $y \sim A e^{\left(-\frac{a}{2} \pm i\Delta\right)x} = A e^{-\frac{a}{2}x} \cos(\Delta x) + B e^{-\frac{a}{2}x} \sin(\Delta x)$

Example: Damped harmonic oscillator.

work out in class.

overdamped, critically damped, underdamped.

Frequently we will encounter linear 2^{nd} order equations with nonconstant coefficients.

Some examples:

1) QM Harmonic Oscillator: $y'' - 2xy' + 2\alpha y = 0$
Hermite's Eqn.

2) Hydrogenic atoms: $xy'' + (1+k-x)y' + \alpha y = 0$
Laguerre's Eqn.

3) Particle in a circular tube:
$$x^2 y'' + x y' + (x^2 - m^2) y = 0$$
Bessel's Eqn.

There are many others. How do we study these?

B. Singular Points and Classifying differential equations.

In general we have $y'' = f(y', y, x)$

If y, y' can be finite and y'' remains finite at x_0, then x_0 is an ordinary point

If y'' diverges from finite y, y' at x_0, then x_0 is a <u>singular</u> point

If we write $y'' + P(x)y' + Q(x)y = 0$

If P, Q are finite at x_0, it is an ordinary point

If P and/or Q diverges at x_0 but

$(x-x_0)P$ and $(x-x_0)^2 Q$ remain finite at $x \to x_0$ then x_0 is a <u>regular</u> or nonessential singular point

Otherwise x_0 is an essential singularity.
Note: You can analyze ∞ by making a change of variables $x \to 1/z$ and looking at $z=0$.

Example: Bessel's Eqn

$$x^2 y'' + x y' + (x^2 - m^2) y = 0$$

$$\Rightarrow P(x) = 1/x \quad Q(x) = 1 - \frac{m^2}{x^2}$$

$\Rightarrow x = 0$ is a regular singular point.
what about ∞?

$$\frac{dy}{dx} = \frac{dy(z^{-1})}{dz}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} = -z^2\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left[-z^2\frac{dy}{dz}\right]\frac{dz}{dx} = \left(-2z\frac{dy}{dz} - z^2\frac{d^2y}{dz^2}\right)(-z^2)$$

$$= 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2} \quad \text{so we have}$$

$$\frac{1}{z^2}\left(2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}\right) + \frac{1}{z}\left(-z^2\frac{dy}{dz}\right) + \left(\frac{1}{z^2} - m^2\right)y = 0$$

$$z^2 \frac{d^2y}{dz^2} + 2z\frac{dy}{dz} - z\frac{dy}{dz} + (z^{-2}-m^2)y = 0$$

so $P(z) = \dfrac{z}{z^2} = 1/z$

$Q(z) = \dfrac{1}{z^4}(1-m^2z^2)$

so $z=0$ $x=\infty$ is an essential singularity.

So, what's the big deal?

Many of the equations of interest have three regular singular points. By appropriate rescaling one can push those points to $0, 1, \infty \Rightarrow$ This is the standard form of the hypergeometric equation

The Bessel equation and a number of others have one regular singular point and one irregular or essential singularity. These can all be mapped onto confluent hypergeometric equation

Fine, but how do you get a solution?

Series Solutions: Frobenius' method

We can develop a series solution about any point that is <u>not</u> an essential singularity.

A linear 2nd order homogeneous DE solution

$y(x) = c_1 y_1(x) + c_2 y_2(x)$ ← We will prove that there are exactly two linearly independent solutions to the homogeneous equation. More on particular solutions later! (We will discuss these when talking about Green's functions)

We now look for one solution in terms of a power series:

An example $y'' + \omega^2 y = 0$ (harmonic oscillator)

Try $y(x) = \sum_{n=0}^{\infty} a_n x^{k+n}$ $(a_0 \neq 0)$

$n \in \mathbb{Z}$ but k need not be.

Well? Let's try it out...

$$y'' = \sum_{n=0}^{\infty} a_n (k+n)(k+n-1) x^{k+n-2}$$ and plugging in...

$$\sum_{n=0}^{\infty} a_n \left[(k+n)(k+n-1) x^{k+n-2} + \omega^2 x^{k+n} \right] = 0$$

For this to vanish we need ~~every~~ the coefficient of every power of x to vanish. We have an infinite set of algebraic equations.

x^{k-2}: $\quad a_0 k(k-1) = 0$

$(n=0)$ $\quad k(k-1) = 0 \implies k=0$ or $k=1$

Indicial Equation ← words that are commonly associated w/ this...

what about for a general n?

$$a_{n+2}[k+n+2][k+n+1] + a_n \omega^2 = 0 \quad \text{or}$$

$$a_{n+2} = -a_n \omega^2 \frac{1}{(k+n+2)(k+n+1)}$$

two term recursion relation

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7$

information propogates forward like so once you have chosen a_0, a_1 (say)

Let's look at the solution for $k=0$ ← see the indicial equation above.

$$a_{n+2} = -\omega^2 a_n \frac{1}{(n+2)(n+1)}$$

$$a_2 = -a_0 \omega^2 \frac{1}{1 \cdot 2}$$

$$a_4 = -a_2\omega^2\frac{1}{4\cdot 3} = (-1)^2 a_0(\omega^2)^2\frac{1}{4\cdot 3\cdot 2\cdot 1}$$

In general we see the pattern: $a_{2n} = \frac{(-1)^n\omega^{2n}}{(2n)!}a_0$

$\Rightarrow$

$(k=0)$ $y(x) = a_0\left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots\right]$

$\rightarrow$ here we set $a_1 = 0$.

$y(x) = a_0\cos(\omega x)$ a solution found by expanding about the ordinary point $x_0 = 0$.

$\star \longrightarrow$ Discuss convergence of the solutions!

Try for yourself: what happens if you don't set $a_1 = 0$?

What happens if you try $k=1$

Ans to part b) you get $\sin(\omega x)$.

Let's now try this on Bessel's Equation.

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

Try $y(x) = \sum_{n=0}^{\infty} a_n x^{k+n}$

$$\sum_{n=0}^{\infty}\left\{a_n(k+n)(k+n-1)x^{n+k} + a_n(k+n)x^{n+k} + a_n x^{n+k+2} - m^2 a_n x^{n+k}\right\} = 0$$

set $n=0$, we get $a_0\left[k(k-1) + k - m^2\right] = 0 \Rightarrow$

$\Rightarrow$ indicial equation $k^2 = m^2 \Rightarrow k = \pm m$

Now what about $n=1$?

$$a_1 [(k+1)(k) + k+1-m^2] = 0$$

$$a_1 [k+1-m][k+1+m] = 0$$

if $k = \pm m$ we must have $a_1 = 0$ (Unless $m = \pm 1/2$
$\Rightarrow$ then we get spherical Bessel functions...)

In general we find $a_{n+2} = -a_n \dfrac{1}{(n+2)(2m+n+2)}$

Solving the recursion problem:

$$a_{2p} = (-1)^p a_0 \frac{m!}{2^{2p} p! (m+p)!}$$

$\Rightarrow$ we have a solution:

$$y(x) = a_0 J_m(x) = a_0 2^m m! \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{m+2j}$$

common name

Note that $J_m(x)$ is even (or odd) whenever m is even (or odd).

What about $k=-m$? if $m \notin \mathbb{Z}$ then we get a perfectly good second solution.

If $m \in \mathbb{Z}$ then what happens?
from $a_{j+2} = -a_j \dfrac{1}{(j+2)(2m+j+2)}$

if $m \in \mathbb{Z}$ and $m<0$ then when $2m=-j-2$ or $j=-2-2m$ the recursion relation blows up! We have no series solution.
See chapter 11 of Arfken → there you will find that $J_{-n} \propto J_n$ i.e. $J_{-n}(x) = (-1)^n J_n(x)$ so only one solution!

So to recap: The score for the series solution is:

$$\left.\begin{array}{l} \text{SHO} \longrightarrow 2 \text{ solutions(!)} \\ \text{Bessel} \rightarrow 1 \text{ solution} \end{array}\right\} \text{This method is too simple to catch all the solutions...}$$

The success depends on the roots of the indicial equation (as we saw w/ Bessel's) and on the degree of the singularity of P and Q.

let's test this in more detail by considering the following equations:

A. $y'' - \frac{6}{x^2} y = 0$; B. $y'' - \frac{6}{x^3} y = 0$

C. $y'' + \frac{1}{x} y' - \frac{a^2}{x^2} y = 0$; D. $y'' + \frac{1}{x^2} y' - \frac{a^2}{x^2} y = 0$

A. $y = \sum_{n=0}^{\infty} x^{n+k} a_n$

$$\sum_{n=0}^{\infty} a_n \{(n+k)(n+k-1)x^{n+k-2} - 6x^{n+k-2}\} = 0$$

indicial equation: $k(k-1) = 6 \Rightarrow k = 3, -2$

but for general n: $x^{n+k-2}\{(n+k)(n+k-1) - 6\}a_n = 0$

$\Rightarrow a_n = 0$ for $n > 0$. $\Rightarrow$ solutions:

But we have two solutions $y = Ax^3 + \frac{B}{x^2}$

B. Now we have an essential singularity at $x_0 = 0$.
What goes wrong here?