* can also expand on rows! Do you see why?

Chapter 4: Determinants, matrices, and group theory.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{or} \quad D = \sum_{i,j,k} \varepsilon_{ijk}\, a_i b_j c_k$$

Levi-Civita symbol

For the 3X3 determinant above we get:

$$D = + a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 +$$

$$+ a_3 b_1 c_2 - a_3 b_2 c_1$$

- Only one term from each (row) column in each product. and all products appear once!
- Note there will be $n!$ terms in the determinant.

Note we can rewrite this as:

$$D = a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

$$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

We have just seen Laplaces "development by minors"

$$D = \sum_{i=1}^{3} (-1)^{i+1} a_i \, |M_{i1}| \overset{\text{in general}}{\Rightarrow} \sum_{i=1}^{3} (-1)^{i+j} A_{ij} \, |M_{ij}| = |A| \; *$$

minor $M_{\alpha\beta}$ ← remove α^{th} row and β^{th} column

- A determinant must change sign if any two rows or columns are exchanged.

$\Rightarrow$ The determinant will vanish if any two rows (columns) are equal.

Note also from Laplace's rule:

$$(\star)\quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + k b_1 & b_1 & c_1 \\ a_2 + k b_2 & b_2 & c_2 \\ a_3 + k b_3 & b_3 & c_3 \end{vmatrix}$$ for any constant k.

Why? Expanding in minors along the 1st column $\Rightarrow$

$$\begin{vmatrix} a_1 + k b_1 & \cdot & \cdot \\ a_2 + k b_2 & \cdot & \cdot \\ a_3 + k b_3 & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} a_1 & \cdot & \cdot \\ a_2 & \cdot & \cdot \\ a_3 & \cdot & \cdot \end{vmatrix} + k \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

↑ This is zero.

The Solution of a set of Homogeneous linear equations

$$a_j x + b_j y + c_j z = 0 \quad \text{for } j = 1, 2, 3$$

Does a nontrivial solution exist? (i.e. one w/o $x = y = z = 0$)

$$x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$$

Now add $y \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + z \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ to the first column using our result at the top of the page ($\star$)

We get:

$$x\begin{vmatrix} & \end{vmatrix} = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = 0$$

↑ But, this column must vanish by the 3 homogeneous, linear equations so

$\Rightarrow x = 0$ or $\begin{vmatrix} & \end{vmatrix} = 0$. $\Rightarrow$ if $x=0$. $y = z = 0$ as well by the same argument.

Thus, either the determinant of the coefficients vanishes or there are no nontrivial solutions.

We have a relationship between determinants vanishing and nontrivial solutions to linear equations.

Review of Matrices:

Representations of <u>linear</u> operators. They act on vectors to generate another vector.

$\vec{v} = A\vec{u}$ ← linearity means we need only know how A acts on a set of basis vectors.

$\vec{u} = u_i \hat{e}_i$ and $A\hat{e}_i = a_{ji}\hat{e}_j$ ← some linear combination of the basis vectors.

↖ Einstein summation convention

Then

$$A\vec{u} = A u_i \hat{e}_i = a_{ji} u_i \hat{e}_j,$$

or if we say $\vec{v} = v_k \hat{e}_k$ then $\boxed{v_k = a_{k\ell} u_\ell}$

We can make a simple picture to remember these rules:

$$a_{ij}$$

row column

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \boxed{v_1} \\ v_2 \\ v_3 \end{pmatrix}$$

multiply row x column to get the row)

Note that the matrix is only a representation of the linear operator A as it depends on the coordinate representation

If two matrices $A = B$ are the same, the operators are equal as well.

We can also add matrices element-by-element

$$A = B + C$$

And multiply by a scalar: $\alpha A = A\alpha$

Matrix multiplication: Just like above, but more to do!

$$AB = C \Longleftrightarrow \quad a_{ij} = b_{ik} c_{kj}$$

Direct Product $\quad A \otimes B = C \qquad C_{\alpha\beta} = A_{ij} B_{kl}$

tensor product

$$\alpha = n(i-1) + k$$
$$\beta = n(j-1) + l$$

Try this out! $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$C = A \otimes B = \begin{pmatrix} 0\cdot 1 & 0\cdot 0 & 1\cdot 1 & 1\cdot 0 \\ 0\cdot 0 & 0\cdot 1 & 1\cdot 0 & 1\cdot 1 \\ -1\cdot 1 & -1\cdot 0 & 0\cdot 1 & 0\cdot 0 \\ -1\cdot 0 & -1\cdot 1 & 0\cdot 0 & 0\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$ Is this familiar to you?

Note what we did was.

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}$$

<u>Diagonal matrices</u> A_{ij} is diagonal $\Longleftrightarrow$ $A_{ij} = 0$ unless $i=j$

For diagonal matrices $AB = BA$; prove it!

<u>The Trace:</u> $\mathrm{Tr}\, A = A_{ii}$ ← sum down diagonals.

$\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$

<u>pf</u>: $\mathrm{Tr}\, AB = A_{i\ell} B_{\ell i} = B_{\ell i} A_{i\ell} = \mathrm{Tr}\, BA$ ✓

Moreover, $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$ cyclic permutation

since $A_{i\ell} B_{\ell j} C_{ji} = C_{ji} A_{i\ell} B_{\ell j} = \mathrm{Tr}(CAB)$

When matrices represent groups of linear operators, the trace of the matrix is called the character. We will see that the trace is invariant under changes of basis due to the cyclic permutation rule above. This will be important later on.

Matrix Inverse: For many operators A an inverse A^{-1} exists $\ni$

$$A^{-1}A = AA^{-1} = I \leftarrow (\delta_{ij})$$

Identity matrix.

The necessary and sufficient conditions for the existence of A^{-1} is $|A| \neq 0$.

Inverting a matrix: Gauss-Jordan Inversion

For 2x2 ones it's easy... $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

just check it.

In general you can do the following:

$$\begin{pmatrix} & A & \end{pmatrix} \qquad \begin{pmatrix} I \end{pmatrix}$$

$\downarrow$ perform identical row operations on each so that $A \to I$, $I \to A^{-1}$

An example:
$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$row_1 - 2row_2$
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$row_3 - 2\,row_1$
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{pmatrix}$$

$row_1 + row_3$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{pmatrix}$$

$row_3 \times -1$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & -4 & -1 \end{pmatrix}$$

resort rows
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & -4 & -1 \\ -1 & 2 & +1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & -4 & -1 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad \checkmark$$

We can try it w/ our 2x2 matrix to see if it works...

Trying the GJ Inversion process on a 2x2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & b/a \\ 1 & d/c \end{pmatrix} \qquad \begin{pmatrix} 1/a & 0 \\ 0 & 1/c \end{pmatrix}$$

$$\begin{pmatrix} 1 & b/a \\ 0 & \frac{d}{c}-\frac{b}{a} \end{pmatrix} \qquad \begin{pmatrix} 1/a & 0 \\ -1/a & 1/c \end{pmatrix} \; ; \quad \frac{d}{c}-\frac{b}{a} = \frac{ad-bc}{ac}$$

$$\begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1/a & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1/a + \frac{bc}{a}\frac{1}{ad-bc} & -\frac{b}{a}\frac{a}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

so the inverse is:

$$\frac{1}{ad-bc}\begin{pmatrix} \frac{ad-bc}{a} + \frac{bc}{a} & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

checks!

Why did this work?

We can define matrices that perform these "row operations" M_L when they left-multiply A.

We did the following $\underbrace{(M_1 M_2 M_3 M_4)}A = I$ Identity matrix.

Successive row operations done in the order 4, 3, 2, 1. In other words, counting backwards.

So $(M_1 \cdots M_n) = A^{-1}$.

To prove this we only need to see how to perform these row operations:

I) Swap rows $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & a \\ b & b \end{pmatrix} = \begin{pmatrix} b & b \\ a & a \end{pmatrix}$

or more generally

swap rows 1 & 3: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix} = \begin{pmatrix} c & c & c \\ b & b & b \\ a & a & a \end{pmatrix}$

Swap rows 1 & 2 $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix} = \begin{pmatrix} b & b & b \\ a & a & a \\ c & c & c \end{pmatrix}$

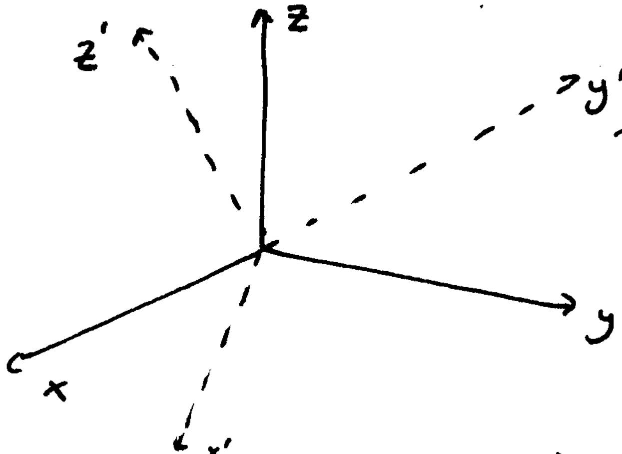
add $p \times$ row 1 to row 2 $\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} a & a \\ b & b \end{pmatrix} = \begin{pmatrix} a & a \\ ap+b & ap+b \end{pmatrix}$

Or, more generally,

add p × row1 to row3 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & 0 & 1 \end{pmatrix} \begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix} = \begin{pmatrix} a & a & a \\ b & b & b \\ ap+c & ap+c & ap+c \end{pmatrix}$

Well, that's all we need!

Orthogonal Matrices and Rotations



Direction cosines: cos(a,b)

$$\hat{x}' = \hat{x}\underbrace{(\hat{x}\cdot\hat{x}')}_{\cos(x,x')} + \hat{y}\underbrace{(\hat{y}\cdot\hat{x}')}_{\cos(y,x')} + \hat{z}\underbrace{(\hat{z}\cdot\hat{x}')}_{\cos(x',z)}$$

$$a_{ab} = \cos(a,b) =$$ cosine of the angle between a and b.

so

$$\begin{matrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{matrix} = \begin{matrix} a_{11}\hat{x} + a_{12}\hat{y} + a_{13}\hat{z} \\ a_{21}\hat{x} + a_{22}\hat{y} + a_{23}\hat{z} \\ a_{31}\hat{x} + a_{32}\hat{y} + a_{33}\hat{z} \end{matrix}$$

Note that $\begin{matrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{matrix} = \begin{pmatrix} & a^T & \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$ Transpose of the a matrix defined in the previous line

* We will change the rules below as a fun sidebar....

(4.11)

Now, $\vec{v} = v_\alpha \hat{e}_\alpha = v_\alpha' \hat{e}_\alpha'$ ← Same vector expressed in two different coordinate systems.

How are the coordinates v_α' and v_α related?

$$\hat{e}_\alpha = a^T_{\alpha\beta}\, \hat{e}_\beta' \quad \text{so}$$

$$v_\alpha \hat{e}_\alpha = v_\alpha a^T_{\alpha\beta} \hat{e}_\beta' = v_\gamma' \hat{e}_\gamma' \Rightarrow$$ Note! We are assuming that both coordinates are orthogonal!! *

$$v_\gamma' = v_\alpha a^T_{\alpha\gamma} = a_{\gamma\alpha} v_\alpha$$

To ensure that the length of the vector $\vec{v}$ is unchanged when we express it in different coordinate systems we must insist that

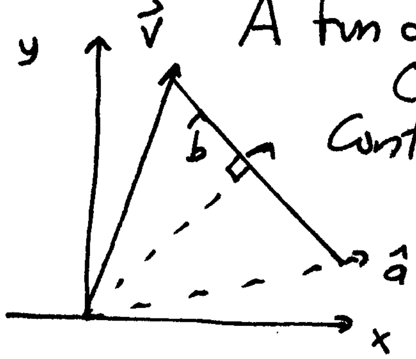
$$|\vec{v}|^2 = |\vec{v}'|^2 \quad \text{or}$$

$$v_\alpha v_\alpha = v_\beta' v_\beta' = a_{\beta i} v_i a_{\beta j} v_j \Rightarrow$$

$$v_\alpha^2 = a_{\beta i} a_{\beta j} v_i v_j \quad \text{for all } \vec{v}.$$

$$\Rightarrow \quad a_{\beta i} a_{\beta j} = a^T_{i\beta} a_{\beta j} = \delta_{ij}$$

or <u>a^T is the inverse of a.</u>



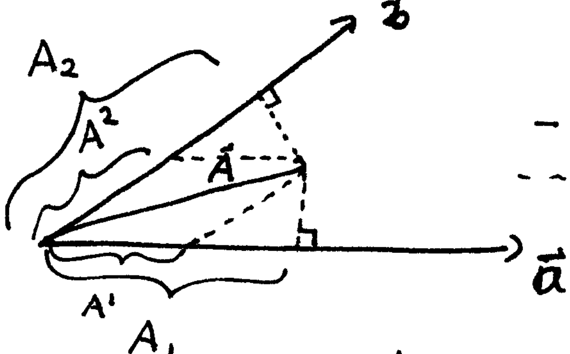
A fun diversion:

Oblique Coordinates, covariant and Contravariant coordinates/vectors.

new basis vectors $\hat{a}, \hat{b}$

$\begin{pmatrix} \hat{a}\cdot\hat{a} \neq 1 \\ \hat{b}\cdot\hat{b} \neq 1 \end{pmatrix}$ and $\hat{a}\cdot\hat{b} \neq 0$

$\vec{V} = v_a \hat{a} + v_b \hat{b}$ and $\vec{V} = v_x \hat{e}_x + v_y \hat{e}_y$



So if we add the vectors $v^a \hat{a} + v^b \hat{b}$ we get $\vec{v}$ by the usual "parallelogram law" of vector addition

we can write $\vec{V} = v^i \hat{e}_i$ where $\hat{e}_1 = \vec{a}$, $\hat{e}_2 = \vec{b}$.

as an example, let's take $\vec{a} = \hat{x}$ and $\vec{b} = b\cos\alpha\,\hat{x} + b\sin\alpha\,\hat{y}$

Then if $\vec{A} = (A_x \hat{x} + A_y \hat{y})$ we have

$$A_x \hat{x} + A_y \hat{y} = A^a \hat{x} + A^b (b\cos\alpha\,\hat{x} + b\sin\alpha\,\hat{y})$$

$$\Rightarrow \left.\begin{aligned} A_x &= A^a + b\cos\alpha\, A^b \\ A_y &= A^b\, b\sin\alpha \end{aligned}\right\} \Rightarrow \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & b\cos\alpha \\ 0 & b\sin\alpha \end{pmatrix}}_{\text{call this } P} \begin{pmatrix} A^a \\ A^b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A^a \\ A^b \end{pmatrix} = \frac{1}{b\sin\alpha} \begin{pmatrix} b\sin\alpha & -b\cos\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

so we get $\begin{pmatrix} A_x \\ A_x b\cos\alpha + b\sin\alpha\, A_y \end{pmatrix} = A_\alpha$.

$\Rightarrow$ the metric tensor "lowers" the indices. It's inverse "raises" them as can be seen by inverting the equation

$g_{\alpha\beta} A^\beta = A_\alpha$. $g^{-1}_{\alpha\beta} = \begin{pmatrix} b^2 & -b\cos\alpha \\ -b\cos\alpha & 1 \end{pmatrix} \dfrac{1}{b^2 - b^2\cos^2\alpha}$

$$g^{-1}_{\alpha\beta} = \frac{1}{\sin^2\alpha}\begin{pmatrix} 1 & -\frac{\cos\alpha}{b} \\ -\frac{\cos\alpha}{b} & b^{-2} \end{pmatrix} = g^{\alpha\beta} \leftarrow \text{then}$$

$g^{\alpha\beta} g_{\beta\delta} = \delta^\alpha{}_\delta \leftarrow$ kronecker delta.

How did we guess the form of the covariant metric tensor $g_{\alpha\beta}$? $g_{\alpha\beta} = \begin{pmatrix} \hat{a}\cdot\hat{a} & \hat{a}\cdot\hat{b} \\ \hat{b}\cdot\hat{a} & \hat{b}\cdot\hat{b} \end{pmatrix}$. But why is this the answer?

To decide this let's look at the length of the vector $|\vec{A}|$.

$$|\vec{A}|^2 = A_x^2 + A_y^2 = \begin{pmatrix} A_x \\ A_y \end{pmatrix}^T \cdot \begin{pmatrix} A_x \\ A_y \end{pmatrix} \leftarrow \text{Recall } (A_x\ A_y)\begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

But $\begin{pmatrix} A_x \\ A_y \end{pmatrix} = P\begin{pmatrix} A^a \\ A^b \end{pmatrix}$ so $|A|^2 = (A^a\ A^b)\, P^T P \begin{pmatrix} A^a \\ A^b \end{pmatrix}$

so it must be that lengths of <u>contravariant</u> vectors are given by $A^i A^j g_{ij}$ and we identify $g = P^T P$

So we have the <u>contravariant</u> form of the vector $\vec{A}$:

$$A^\alpha = \begin{pmatrix} A_x - \cot\alpha\, A_y \\ A_y / b\sin\alpha \end{pmatrix}$$

We can also find the <u>covariant</u> form of the vector $\vec{A}$:

$$A_1 = \vec{a}\cdot\vec{A} = A_x \qquad A_2 = \vec{b}\cdot\vec{A} = A_x b\cos\alpha + A_y b\sin\alpha$$

$$A_\alpha = \begin{pmatrix} A_x \\ A_x b\cos\alpha + A_y b\sin\alpha \end{pmatrix}$$

We define a new tensor: (metric tensor). If P is defined as on the previous page, we write

$$g = \tilde{P}P \Rightarrow g = \begin{pmatrix} 1 & b\cos\alpha \\ b\cos\alpha & b^2 \end{pmatrix}$$

$$g_{\alpha\beta}A^\beta = A_\alpha \quad \text{check.} \quad \begin{pmatrix} 1 & b\cos\alpha \\ b\cos\alpha & b^2 \end{pmatrix}\begin{pmatrix} A_x - \cot\alpha\, A_y \\ A_y/b\sin\alpha \end{pmatrix} = \begin{pmatrix} A_x - \cot\alpha\, A_y + \cot\alpha\, A_y \\ \cancel{A_x b\cos\alpha - b\sin\alpha\, A_y + \frac{b A_y}{\sin\alpha}}\; A_y \end{pmatrix}$$

$$= \begin{pmatrix} A_x \\ A_x b\cos\alpha - \dfrac{b\cos^2\alpha}{\sin\alpha}A_y + \dfrac{bA_y}{\sin\alpha} \end{pmatrix}$$

$$\text{but} \quad -\frac{\cos^2\alpha}{\sin\alpha} + \frac{1}{\sin\alpha} = \frac{1-\cos^2\alpha}{\sin\alpha} = \frac{\sin^2\alpha}{\sin\alpha} = \sin\alpha$$

4.15

Now, can we write this in terms of the basis vectors $\vec{a}, \vec{b}$?

Recall $P = \begin{pmatrix} \hat{x}\cdot\vec{a} & \hat{x}\cdot\vec{b} \\ \hat{y}\cdot\vec{a} & \hat{y}\cdot\vec{b} \end{pmatrix}$ so that

$$P^T P = \begin{pmatrix} \hat{x}\cdot\vec{a} & \hat{y}\cdot\vec{a} \\ \hat{x}\cdot\vec{b} & \hat{y}\cdot\vec{b} \end{pmatrix}\begin{pmatrix} \hat{x}\cdot\vec{a} & \hat{x}\cdot\vec{b} \\ \hat{y}\cdot\vec{a} & \hat{y}\cdot\vec{b} \end{pmatrix} = \begin{pmatrix} (\hat{x}\cdot\vec{a})^2+(\hat{y}\cdot\vec{a})^2 & (\hat{x}\cdot\vec{a})(\hat{x}\cdot\vec{b}) + (\hat{y}\cdot\vec{b})(\hat{y}\cdot\vec{a}) \\ & (\hat{x}\cdot\vec{b})^2+(\hat{y}\cdot\vec{b})^2 \end{pmatrix}$$

Now $(\hat{x}\cdot\vec{a})^2+(\hat{y}\cdot\vec{a})^2 = |a|^2 = \vec{a}\cdot\vec{a}$ same for $\vec{b} \Leftrightarrow \vec{a}$.

What about $(\hat{x}\cdot\vec{a})(\hat{x}\cdot\vec{b}) + (\hat{y}\cdot\vec{a})(\hat{y}\cdot\vec{b}) = \vec{a}\cdot\vec{b}$

so $$g = \begin{pmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \end{pmatrix}$$

Finally, why did we never worry about the distinction between covariant and contravariant vectors before? Ans. We took orthonormal coordinate systems!

if $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \Rightarrow g_{ij} = \delta_{ij}$ if we go back to our previous example we see that when we take $b=1$ and $\alpha = \pi/2$ we get

$$A^\alpha = \begin{pmatrix} A_x \\ A_y \end{pmatrix} = A_\alpha \quad \text{and } g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Things are back to our simpler case!

Diagonalization, similarity transformations, eigenvalues and eigenvectors.

<u>Similarity Transformation</u> Let A be linear operator in a (vector) space spanned by $\hat{e}_1, \ldots, \hat{e}_n$. Its action on any vector $A_i \hat{e}_i$ can be written in terms of a matrix A_{ij} acting on the column vector of components A_i i.e.

$$\underline{\underline{A}} \cdot \vec{B} = \vec{C} \Rightarrow C_i = A_{ij} B_j$$

What if we want to express this operator in another basis? Say that the matrix $\underline{\underline{S}}$ performs the change of basis

$$\underline{\underline{S}} \cdot \hat{e}_i = \hat{e}_i' .$$ Then $\underline{\underline{S}} \cdot \vec{B} = \vec{B}'$ ← same vector in the new basis

In other words $S_{ij} B_j = B_i'$

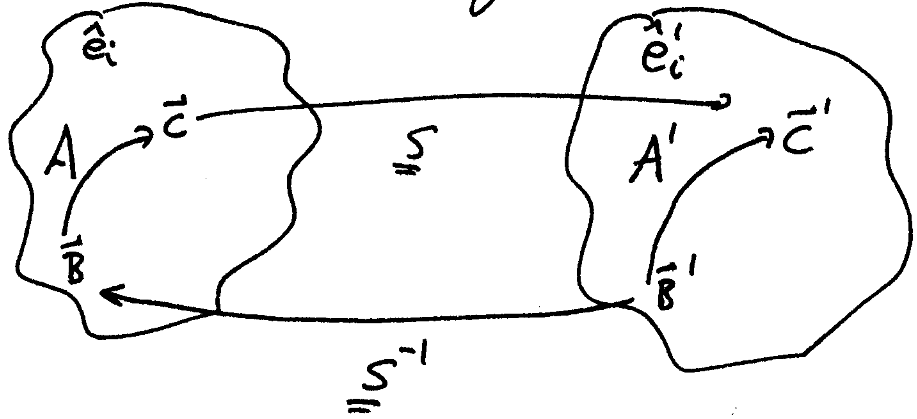
Now, how does $\underline{A}'$ act in the new basis, or more precisely what matrix represents $\underline{\underline{A}}$ in the new basis?

$$\underline{\underline{A}}' \cdot \vec{B}' = \vec{C}' \Rightarrow \underline{\underline{A}}' \cdot (\underline{\underline{S}} \cdot B) = \underline{\underline{S}} \cdot C \Rightarrow$$

$$(\underline{\underline{S}}^{-1} \cdot \underline{\underline{A}}' \cdot \underline{\underline{S}}) \cdot \vec{B} = \vec{C} \quad \text{so} \quad \underline{\underline{S}}^{-1} \underline{\underline{A}}' \underline{\underline{S}} = A$$

$$\Rightarrow \boxed{\underline{\underline{A}}' = \underline{\underline{S}} \cdot \underline{\underline{A}} \cdot \underline{\underline{S}}^{-1}}$$

How to think about this? Use S to let A always act in the basis that we already understand.



so $A' = S A S^{-1}$. The picture makes it clear.

Is it possible to find a similarity transform that will bring a matrix into diagonal form? If so for what matrices will this work and how can we do this?

If the matrix representing the operator A is diagonal then

eigenvector $\longrightarrow$ $\underline{A} \cdot \vec{r}_i = \lambda_i \vec{r}_i$ for a set of $i = 1, \ldots, d$ vectors where (no sum) eigenvalues.

d is the dimension of the vector space.

Is this possible? After all, this means that

$(\underline{A} - \lambda_i \underline{\underline{I}}) \cdot \vec{r}_i = \vec{0}$ for a vector $\vec{r}_i$ that is <u>not</u>

zero. Thus $\underline{A} - \lambda_i \underline{\underline{I}}$ must be rendered non-invertable

for the appropriate choice of the value λ_i.