Rheology and Contact Lifetimes in Dense Granular Flows

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The rheology and distribution of interparticle contact lifetimes for gravity-driven, dense granular flows down an inclined plane are studied using large-scale, three dimensional, granular dynamics simulations. We show that for cohesionless particles, rather than observing a large number of long-lived contacts as might be expected for dense flows, binary collisions of the minimal possible duration predominate. In the hard particle limit, the rheology conforms to Bagnold scaling, where the shear stress is quadratic in the strain rate. As the particles are made softer, however, we find significant deviations from Bagnold rheology; the material flows more like a viscous fluid. We attribute this change in the collective rheology of the material to subtle changes in the contact lifetime distribution involving the emergence of an increasing number of long-lived contacts in the system.

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The rheology of granular materials is relevant to many areas of nature and industry, from mountain avalanches and mud-slides, to grain transport and storage [1, 2]. A particularly simple class of granular flow, which continues to be studied, is gravity-driven, dense granular flow down a rough, inclined plane. This geometry is the archetypal granular flow with which one can study the relation between the stress state and the dynamics and structure, *i.e.* constitutive relation of the granular material. Indeed, a number of recent well-controlled experiments [3–6], and large-scale simulations [7–11], have motivated numerous theories that capture some of the features of inclined plane flows [12–19].

To date, however, the most generally accepted treatment of granular rheology is still the physical picture put forward by Bagnold over 50 years ago [20–22]. Bagnold rheology describes a mechanism of momentum transfer between particles in adjacent layers, and assumes instantaneous binary collisions between the particles during the flow. Under these assumptions, the inverse strain rate is the only relevant time scale in the problem leading to a constitutive relationship between the shear stress σ and strain rate $\dot{\gamma}$ of the form

$$\sigma = \kappa\dot{\gamma}^2, \tag{1}$$

where κ is independent of $\dot{\gamma}$.

Despite recent concerns on the validity of Bagnold's original experiments [23], and the applicability of the theory [24, 25], Eq. 1 has proved rather successful in predicting the rheology of dense granular flow down an inclined plane [3, 8]. However, this is somewhat add odds to the idea that Bagnold theory should apply only to rapid flows of hard-spheres where binary collisions predominate. By their very nature, dense flows [26] are thought to be controlled by enduring, multiple interparticle contacts forming extended stress-bearing structures [12] and/or large length-scale cooperative dynamics [17, 27, 28]. Previous work [8] revealed hat the stresses arising from the contact forces are practically an order of magnitude larger than the kinetic stresses generated via velocity fluctuations.

To provide further insight into this problem, we analyze the interparticle contact dynamics of dense flows and relate these to the bulk rheology. We find that lifetimes of two contacting particles are of the same order of magnitude as the binary collision time and insensitive to the location of the pair within the flowing pile. In contrast, the strain rate is strongly dependent on the height and varies considerably over the parameter space studied here. The scenario that emerges, in some sense goes against the grain: Even though the flows are dense, the dynamics at the microstructural scale remain dominated by short-time collisions. In the hard-particle limit, when the inverse strain rate - the shear time - is much longer than the collision time, Bagnold theory, Eq. 1, accurately describes the flow. When the particles are soft, the Bagnold relation must be supplemented by an additional term linear in the strain rate. The linear term appears to result from the creation of a small population of long-lived interparticle contacts (comparable to the shear time). One of the principal results of this work is that the functional form of the constitutive law is sensitively dependent on changes in the long-time tail of the interparticle contact time distribution.

Our simulations are based on the model developed by Cundall and Strack [29], and Walton [30], and has been shown to quantitatively match experimental data [3, 8, 31]. We study $N = 35,900$ monodisperse spheres of diameter d and mass m, flowing on a rough base of length $20d$, and width $20d$, tilted an angle θ with respect to gravity. We use periodic boundary conditions in the flow and vorticity directions. The height h, of the flowing pile is between $90d < h < 100d$ depending on the angle of inclination. The inelastic contact interactions are modeled as a spring-dashpot for forces both normal (n) and tangential (t) to the interparticle contact plane. For details see Ref. 8. The parameters of most relevance in this study are $k_{\rm n,t}$ and $\gamma_{\rm n,t}$, the elastic and dissipative constants for interparticle interactions. At existing contacts we include a friction model that obeys the Coulomb yield criterion while the non-dissipative normal forces are modeled by Hookean springs. Two particles are identified as contacting neighbors when their surfaces overlap. Similar results were found for Hertzian contacts [32]. For Hookean contacts the coefficient of restitution parameterizes the dissipative nature of the interparticle collisions and is given by $e_{\rm n} = \exp\left(-\gamma_{\rm n} t_{\rm col}/2\right)$, where $t_{\rm col}$ is the binary collision time,

$$t_{\rm col} = \pi[2k_{\rm n}/m - \gamma_{\rm n}^2/4]^{-1/2}. \tag{2}$$

We kept $e_{\rm n} = 0.88$ fixed throughout this study, while varying $k_{\rm n}$ and $\gamma_{\rm n}$. We set $k_{\rm t} = 2k_{\rm n}/7$ and $\gamma_{\rm t} = 0$. For example, for $k_{\rm n} = 2\times10^5 mg/d$, $\gamma_{\rm n} = 50\tau^{-1}$, where $\tau = \sqrt{d/g}$. The coefficient of friction is $\mu = 0.5$. The time step for all simulations with $k_{\rm n} \leq 2\times10^5 mg/d$ was $\delta t = 10^{-4}\tau$. For $k_{\rm n} = 2\times10^6 mg/d$, $\delta t = 2.5\times10^{-5}$, for $k_{\rm n} = 2\times10^7 mg/d$, $\delta t = 10^{-5}$ and for $k_{\rm n} = 2\times10^9 mg/d$, $\delta t = 10^{-6}$. After a steady state had been achieved (up to six billion time steps for the largest stiffness), we collected lifetime data for runs of several million time steps.

In Fig. 1(a) we show the distributions, $P(\tau_c^* \equiv \tau_c/t_{\rm col})$, of two-particle contact lifetimes, τ_c, normalized by the binary collision time, $t_{\rm col}$ for spring constants ranging from $k_n = 2\times10^3 - 2\times10^9 mg/d$. Under this time rescaling all distributions exhibit a prominent short-time peak near the binary collision time $\tau_c^* = 1$, and an approximately exponential decay towards longer contact lifetimes. As $k_{\rm n}$ is reduced this exponential tail becomes broader indicating an increasing density of enduring contacts. The data shown in Fig. 1 is depth-averaged over contacts in the flowing pile away from the bottom boundary and the top saltating layer.

The normalized mean contact lifetime exhibits essentially no depth dependence as shown in Fig. 2(a). Given the stress–free boundary condition at the free surface of the pile, $\dot\gamma \to 0$ there so the normalized inverse strain rate, $\dot{\gamma^*}^{-1} \equiv \dot\gamma^{-1}/t_{\rm col}$, shown in Fig. 2(b) must be height dependent. It also depends strongly on k_n, varying by several orders of magnitude in our data set.

As shown in Fig. 3, over several orders of magnitude in $k_{\rm n}$, the mean normalized contact time remains nearly constant while the maximum contact time $\tau_{c_{\rm max}}$, extracted from the distributions in Fig. 1, decreases with increasing stiffness, reflecting the narrowing of the contact time distributions as the particles become harder. The average number of contacting neighbors per particle $n_{\rm c}$, shown in Fig. 3(b), tends to the binary collision limit ($= 0$) as the particles become harder.

Based on these data one might expect that the Bagnold constitutive law holds in the system as the duration of typical interparticle contacts is small compared to the shear time allowing one to imagine that momentum transport is dominated by effectively instantaneous

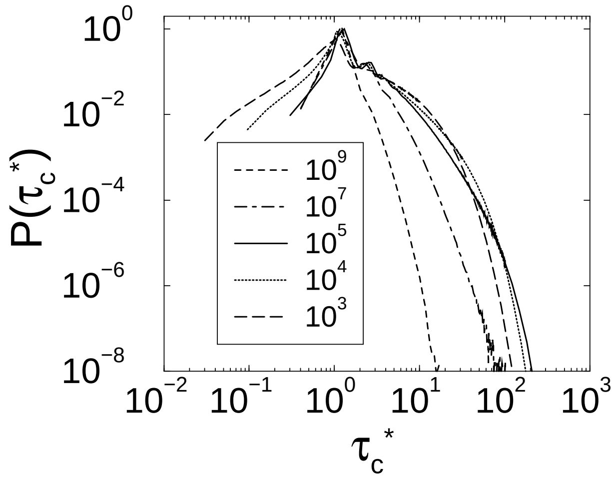

FIG. 1: Dependence of the distributions $P(\tau_c^*)$ of the depth-averaged normalized contact lifetimes, $\tau_c^* \equiv \tau_{\rm c}/t_{\rm col}$, on particle stiffness $k_{\rm n} = 2\times[10^9,\ 10^7,\ 10^5,\ 10^4,\ 10^3] mg/d$. Flow at $\theta = 23°$.

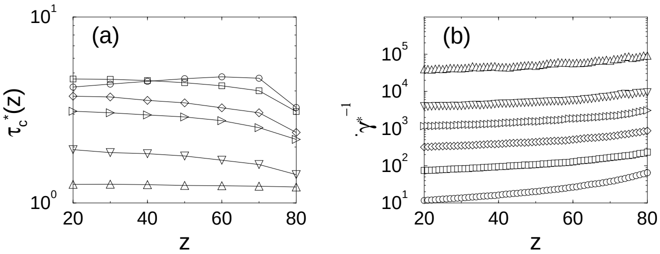

FIG. 2: Depth profiles of, (a) the normalized contact time τ_c^* (line is a guide to the eye) and, (b) the normalized shear time $\dot{\gamma^*}^{-1} \equiv \dot\gamma^{-1}/t_{\rm col}$, for different particle stiffness, $k_{\rm n} = 2\times [10^9\ (\triangle),\ 10^7\ (\triangledown),\ 10^6\ (\triangleright),\ 10^5\ (\Diamond),\ 10^4\ (\Box),\ 10^3(\circ)] mg/d$. $z = 0$ defines the bottom of the pile. Flow at $\theta = 23°$.

binary collisions.To test this we now turn to a characterization of the granular rheology by fitting the velocity profile of the flowing material to the prediction of a modified Bagnold relation [33] of the form

$$\sigma = \kappa\dot\gamma^2 + \beta\dot\gamma, \tag{3}$$

where the coefficients κ and β are determined by a least-squares fit to the velocity data. To characterize the deviations from standard Bagnold rheology consider the ratio of the shear stress in the Bagnold and linear forms: $\Omega \equiv \beta/\kappa\dot\gamma$. Ω measures the competition between momentum transfer through binary collisions (κ) and enduring contacts (β).

Figure 4 shows the dependence of Ω, on particle stiffness for a system flowing at $\theta = 23°$. In the hard-particle limit we expect $\Omega \longrightarrow 0$. This is practically achieved already for $k_{\rm n} \geq 2\times10^5 mg/d$, indicating that this value of $k_{\rm n}$ is appropriate for modeling hard (glass) particles, validating this choice in earlier work [8, 31]. As the particles are made softer, Ω grows and the constitutive law

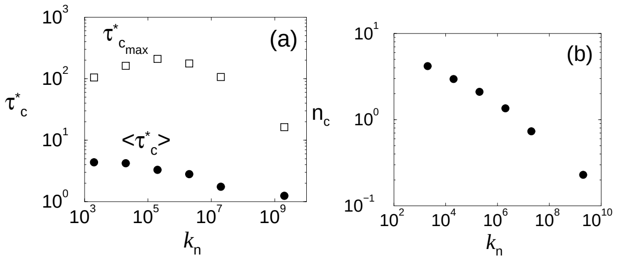

FIG. 3: Dependence on the particle stiffness k_n, of, (a) the depth-averaged contact lifetime $< \tau_c^* >$ (filled circles) and maximum contact time $\tau^*_{c_\mathrm{max}}$ (open squares), both normalized by t_col, and (b) the depth-averaged number of contact neighbors per particle, n_c. Flow at $\theta = 23°$.

approaches a linear relation reminiscent of a viscous fluid. Since Ω is inversely dependent on $\dot{\gamma}$, it is no surprise that it grows monotonically as one approaches the free surface as is shown by the inset to Fig. 4. We have examined the (weaker) dependence of Ω on all other particle interaction parameters, *e.g.* friction and inelasticity; these results will be discussed elsewhere [32].

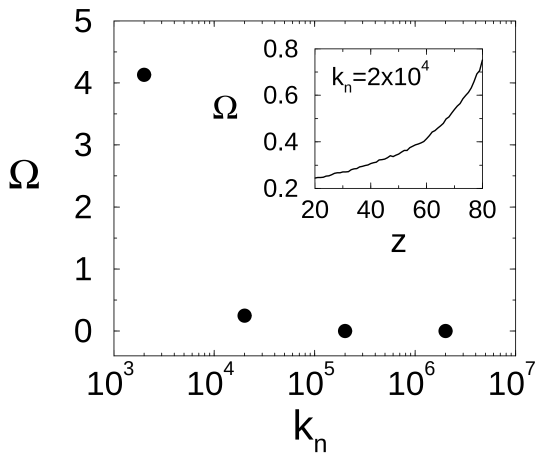

FIG. 4: Dependence of the parameter Ω on the stiffness k_n for $\theta = 23°$. Taken from the middle of the pile at $h/2$. Inset: Height dependence of Ω, for $k_\mathrm{n} = 2 \times 10^4$.

The picture that emerges is that even for the dense flows, as those discussed here, the contact dynamics of *all* systems studied are dominated by binary collisions. The rheology of these granular systems, however, changes dramatically with particle hardness going from the quadratic in shear rate Bagnold law for hard particles to a nearly linear in shear rate or viscous relation for softer particles. The dramatic change in the constitutive relation appears to be controlled by a more subtle feature of the contact time distribution than simply the mean value. In all our data, the mean contact lifetime remains dominated by frequent, rapid, binary contacts that endure for no more than t_col, such that $\langle \tau_\mathrm{c} \rangle \dot{\gamma} \ll 1$. Nevertheless there is a significant change in the rheology of the granular medium as the system goes from obeying the Bagnold relation to acting like a viscous fluid upon reducing particle hardness. This rheological change appears to be due to the growth of the width of the contact time distribution reflecting the appearance of more long-lived contacts in the softer systems. In all cases, however, the size of this population of enduring contacts remains small in comparison to the more common short-lived contacts with $\tau_\mathrm{c} \sim t_\mathrm{col}$.

We suspect that growth of a small population of long-lived contacts leads to the formation of transient stress-bearing structures within the flowing material. These structures span streamlines in the flow and thus elastically transmit stress across their length in proportion to the rate of particle impacts with these structures $\sim \dot{\gamma}$. This reasoning suggests that the shear rate is the appropriate local clock with which to measure contact lifetimes τ_c so that the dimensionless contact lifetime $\tau_\mathrm{c}\dot{\gamma}$ is the fundamental quantity controlling deviations from Bagnold rheology in the flowing state. Given similar contact time distributions, faster flows should then deviate more strongly from the Bagnold prediction than slower flows. This is similar to the conclusions of Campbell [34, 35] for sheared systems.

To explore this point, we use the inclination angle of the pile θ to adjust the overall shear rate and study the resulting change in the observed rheology as parameterized by Ω. In Fig. 5 we plot Ω vs θ for a system of soft ($k_\mathrm{n} = 2 \times 10^4$mg/d) particles where we expect to see generically pronounced deviations from the Bagnold constitutive relation. Indeed, Ω increases with increasing angle. The contact time distribution is less strongly dependent on θ so the principal effect of the changing inclination angle is to shift all the dimensionless contact lifetimes to larger values as the fundamental clock-rate $\dot{\gamma}$ of the system increases. An examination of the contact lifetime distribution allows us to rationalize this somewhat counter-intuitive result that rapid flows conform less to the Bagnold model.

In summary, we have provided extensive simulation results on the rheology and interparticle contact statistics of gravity-driven, dense, granular flows. We observe a transition from a Bagnold-type constitutive relation to one reminiscent of a Newtonian fluid as the particles are made softer. This work addresses the question of relating the local collisional dynamics of the particles at the microscale to the collective rheology the many-body system. Based on our numerical data, we suggest a generalized Bagnold relation [33] to account for this transition and to quantify the changing rheology. Furthermore, despite the naïve guess that the flow in such dense systems should be controlled by long-lived, stress-bearing structures, it turns out that for both hard and soft particles the microscopic particle dynamics is dominated by frequent, short-time, binary collisions. When examining the entire contact lifetime distribution, however, we note

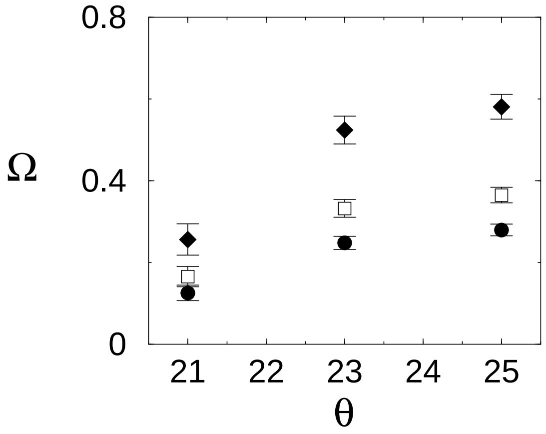

FIG. 5: Dependence of Ω, on inclination angle θ, at different heights inside the pile (h=top of pile): $\frac{h}{4}$ (●), $\frac{h}{2}$ (□), and $\frac{3h}{4}$ (♦) for steady state flow for $k_{\rm n} = 2 \times 10^4 mg/d$.

that as the constituent particles are made softer there is a growth of the width of the contact lifetime distribution due to the emergence of more long-lived contacts as measured in units of inverse shear rate. We propose that the emergence of these atypically long-lived contacts is related to the dramatic change in the granular rheology with particle stiffness and thereby rationalize the surprising result that larger inclination angles, and hence faster flows are actually less Bagnold in rheology than slower flows at smaller inclination angles. It appears that the constitutive relation of granular media interpolates between these extremes and is controlled by a combination of the particle hardness and flow rate. It remains still to ask if other properties of the interparticle interaction modify the collective rheology of the system as strongly as does the particle hardness.

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