1. **Probability: Two random walkers**

Two random walkers in one dimension (the $x$ axis) start at $x = 0$. Each takes steps of $\pm 1$ simultaneously once per second. What is the probability that the walkers are at the *same place* after $N$ seconds (i.e. $N$ steps)? You may assume that both walkers are unbiased so that their probabilities of taking a left and a right step are both equal to $1/2$.

**Hint:** You may find it helpful to think about the random walk of the distance between the two walkers!

**Extra Credit:** See if you can solve the generalized version of this problem where the walkers are biased. You may assume that both have a probability of a rightward step of $p$ and a leftward step of $q$. You may leave your answer in terms of $p, q, N$, and a *single remaining sum* if you need to.

2. **Competition and natural selection: the distribution of the largest values from a random process**

You are investigating a number of well separated bacterial colonies. You find that the fitness of members of each colony, as measured by some variable $x$ and compute the distribution of that variable $x$ over a large ensemble of noninteracting colonies. You wish to test the following hypothesis: Each colony started out with bacterial having a range of fitness variables
\( \{r_1, r_2, \ldots, r_n\} \) with \( r_i \in [0, 1] \). However, after some time only the fittest bacteria survived due to the fact they competed for food. So for each colony the fitness value that you measure \( x \) is related to the initial distribution of fitnesses \( r_i \) by \( x = \max \{r_1, r_2, \ldots, r_n\} \).

(a) In order to test your this theory, compute the probability distribution for the maximum value \( x \) of \( n \) values \( r_i \) selected independently from a probability distribution \( p(r) \). You may leave your answer in terms of \( p(r) \).

(b) Now, to simplify your model further, you assume that \( p(r) = \text{const.} \) In other words, each value of \( r \) was equally likely at the founding of the colony. Calculate the mean and variance of \( x \) as a function of \( n \). Comment on the behavior of these values for large \( n \), i.e. the case in which each bacterial population was founded by a large number of bacteria.

3. A directed random walk

The motion of a dust mote in the air can be considered to be a series of independent steps of length \( \ell \). Due to vertical air currents in the room, each step makes an angle \( \theta \) with the \( z \) axis with probability density

\[
p(\theta) = \frac{2}{\pi} \cos^2(\theta/2),
\]

while the azimuthal angle is uniformly distributed between 0 and \( 2\pi \). Note that the solid angle factor of \( \sin \theta \) is already included in the definition of \( p(\theta) \), which is normalized to one. The dust mote starts at the origin and makes a large number of steps \( N \). Calculate the expectation values \( \langle x \rangle, \langle y \rangle, \langle z \rangle, \langle x^2 \rangle, \langle y^2 \rangle, \langle z^2 \rangle \). Explain what your answer means!

4. Approximating the binomial distribution in the case of rare events: the Poisson distribution

You now know that the probability \( P(n) \) that an event characterized by a probability \( p \) occurs \( n \) times in \( N \) trials is given by

\[
P(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}.
\]

Consider a situation where the probability \( p \) is very small so that the probability is appreciable only for \( n \ll N \). You can now make a number of approximations:

- You may let \( \ln(1-p) \approx -p \). Use this to simplify the factor of \( (1-p)^{N-n} \).
- Show that \( \frac{N!}{n!(N-n)!} \approx N^n \) in this limit.

Show that, using these approximations, you may reduce the binomial distribution given above to the Poisson distribution given by

\[
P(n) = \frac{\lambda^n}{n!} e^{-\lambda}
\]

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5. Two-state systems and the microcanonical ensemble

Consider an isolated system of \( N \gg 1 \) weakly interacting spins with \( s = 1/2 \). Each spin one-half particle has a magnetic moment \( \mu \) which can point either parallel (up) to antiparallel (down) to an applied magnetic field \( H \). The energies of the up and down states are respectively \( -\mu H \) and \( \mu H \). Let \( n_+ \) be the number of up spins and \( n_- \) be the number of down ones.

(a) Calculate the energy \( U \) of the system in terms of the variables given above.

(b) Consider an energy range between \( U + \delta U \) and \( U \), where \( \delta U/U \ll 1 \) but \( \delta U \gg \mu H \). Write down the total number of states in this energy range \( \Omega(U) \).

(c) Write down an expression for \( \ln \Omega(U) \) as a function of \( U \). You should simplify your result using Stirling’s formula.

(d) If \( U \) is far from its maximum or minimum values, use the Gaussian approximation to obtain a simple expression for \( \Omega(U) \)

(e) Compute the temperature of your system recalling \( \beta = \partial \ln \Omega/\partial U \). Find the relation between absolute temperature \( T \) and total energy \( U \) for this system.

(f) Under what circumstances is \( T < 0 \)?